Economics Department

Working Papers in Economics

Boston College *Year* 1998

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September 1998

Abstract

This paper investigates property tax systems (linear taxes on pre-development land value, post-development structure value, and post-development site value) from a partial equilibrium perspective. Particular attention is paid to characterizing property tax systems that are neutral with respect to the timing and density of development and to calculating the deadweight loss from non-neutral property tax systems.

JEL code: H2

Keywords: property taxation, site value taxation, land taxation, neutrality

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* I would like to thank seminar participants at the Universities of Alberta, British Columbia, Calgary, California-Berkeley, Cergy-Pontoise, Oldenberg and New Hampshire for helpful comments, and Dennis Capozza, Yoshitsugu Kanemoto, Geoffrey Turnbull, and Helen Ladd for updating me on the literature.

<u>Neutral Property Taxation</u>

Suppose, for the sake of argument, that a parcel of land has an “intrinsic value” that is unaffected by decisions concerning its current use. A tax on such intrinsic value would be neutral — would not affect decisions concerning its current or future use. This principle has led many economists through the years to advocate the use of land value (or, synonymously, site value) taxation, and the replacement of the current non-neutral property tax system with a land value tax system.

The obvious difficulty is to come up with a definition of land value that is not only neutral, but also fair, practicable and sensible. If the market value of a plot of land depends only on the “attractiveness” of its location (in contrast to the quality of servicing of the land, for example), a tax on the market value of land is neutral. Even granting this condition, there is an unavoidable problem. Because of the durability and immobility of structures, there is no “market” value for developed land. The value observed in the market for a developed site is its property value. And there is no economically correct way to decompose this value into land value and structure value.

During the 1970’s four papers (Shoup (1970), Skouras (1978), Bentick (1979), and Mills (1981)) independently examined arguably the most intuitive decomposition, defining post-development site value as property value minus the depreciated cost of the structure on the site. This definition, here termed *residual site value*, is appealing because it is intuitive and would be relatively easy to implement for tax purposes. The results of these papers can be obtained from a simple model of a developer who owns a unit area of vacant land. He must decide, under perfect foresight, when to develop the property and at what density. A land or property tax system is said to be neutral if its application does not alter the developer’s timing or density decisions. What Shoup et al. (who collectively shall be referred to as *the revisionists*) showed was that the taxation of residual site value is non-neutral; in particular, it discourages density. This result received widespread attention since it called into question the conventional wisdom concerning the neutrality of land taxation.

Subsequent work (Tideman (1982)) has shown that neutrality *is* achieved when post-development site value is defined as “what the site would be worth if there were no structure on it” — here termed *raw site value* — since this value is unaffected by the developer’s decisions. Use of this hypothetical value has the disadvantage, however, that it cannot be simply calculated or inferred on the basis of market observables. Thus, it would appear that the choice of definition of post-development site value for site value taxation purposes entails a tradeoff between deviations for neutrality and considerations of administrative feasibility.

This paper asks whether it is not possible to get the best of both worlds — to avoid the tradeoff — with a well-chosen *property* tax system. More specifically, with separate tax rates on pre-development land value, post-development *residual* site value, and structure value, is neutrality achievable? Since there are three objectives — neutrality with respect to development timing, neutrality with respect to development density, and expropriation of a desired fraction of value — and three instruments, a positive answer is plausible. At least for the model employed — which assumes perfect competition and no uncertainty, among other things — the paper proves that there is indeed a neutral property tax system which employs the residual definition of site value. This positive result provides a basis for optimism in the search for a property tax system that is both practicable and close to neutral. The paper goes on to derive the tax rates that achieve neutrality for the special case where post-development rents grow at a constant rate and vacant land generates no rent: Pre-development land value is untaxed, post-development residual site value is taxed at a rate chosen to meet the revenue requirement, and structure value is *subsidized*. The paper also provides the first — to my knowledge — derivation of the deadweight loss from non-neutral site and property value taxation.

Section I sets the stage by providing a detailed synthetic review of the literature. Section II presents the analytical results concerning neutral property taxation, and briefly discusses some of the problems that would be encountered in moving from theory to practice. Section III derives the deadweight loss from non-neutral property tax systems. Section IV summarizes and concludes.

I. Setting the Stage

To begin, a few words on terminology are appropriate. First, throughout the paper, the terms land value and site value are employed completely interchangeably. Second, a distinction is made between a land value tax system and a property tax system. A land or site value tax system taxes only land or site value and at the same rate before and after development. According to the usage in the paper, the basis for land taxation prior to development is simply the market value of the vacant land. After development, however, when there is a durable and immobile structure on the site, there are not separate *market* values for the site and the structure. Site value is then an abstract or hypothetical notion; it must be imputed. As we shall see, whether site value taxation is neutral hinges on the definition of post-development site value employed. A property tax system, meanwhile, is characterized by three tax rates: a tax rate on the market value of vacant land which applies prior to development, and separate post-development tax rates on site value and structure value. Thus, according to this terminology, a land value tax system is a special case of a property tax system.

Under a site or land value tax system, the same tax rate is applied to land value before and after development. This need not be the case under a property tax system. Accordingly, the site value tax rate under a land value tax system is the tax rate applied to both pre-development land value (a market value) and post-development land value (a hypothetical value). When discussing property tax systems, a distinction must be made between the tax rate on pre-development and post-development land value. The former is referred as the tax rate on vacant land and the latter as the site value tax rate.

I.1 Synthesis of the Literature on the Taxation of Land

The previous literature has employed a variety of models. All the qualitative results obtained can, however, be illustrated using variants of the Arnott-Lewis (1979) model of the transition of land to urban use. An atomistic landowner owns a unit area of undeveloped land. He must decide when to develop the land and at what density to build the structure. Once built, the structure is immutable; no depreciation occurs and no redevelopment is possible. He makes his decision under perfect foresight (and hence under no uncertainty).

To start, consider the landowner-developer's problem in the absence of taxation. The following notation is employed:

t time (t=0 today)

T development time

K development density (the capital-land ratio)

Q(K)	structure production function $(Q' > 0,\ Q'' < 0)$
r(t)	rent per unit of structure at time t
i	interest rate
p	price per unit of structure capital

The structure production function indicates how many units of structure are produced when K units of capital are applied to the unit area of land. For concreteness, one may think of Q as the number of units of rentable floor area per unit area of land (the floor-area ratio), or the number of storeys in the building on the site. The interest rate, the price per unit of capital, and the structure production function are assumed invariant over time to simplify the analysis.

Under the simplifying assumption that land prior to development generates no rent, the developer's problem in the absence of taxation is

$$\max_{T,K} \quad \Pi(T,K) = \int_T^\infty r(t)Q(K)e^{-it}dt - pKe^{-iT} \tag{1}$$

The first-order conditions are

$$T: \quad \left(-r(T)Q(K) + ipK\right)e^{-iT} = 0 \tag{2}$$

$$K: \quad \left(\int_T^\infty r(t)Q'(K)e^{-i(t-T)}dt - p\right)e^{-iT} = 0. \tag{3}$$

Eq. (2) states that, K fixed, development time should be such that the marginal benefit from postponing construction one period (the one-period opportunity cost of construction funds) equal the marginal cost (the rent foregone). Eq. (3) states that, T fixed, capital should be added to the land up to the point where the increase in rental revenue due to an extra unit of capital, discounted to development time, equal the cost of the unit of capital. Figure 1 plots (2) and (3) in T-K space[1]. At a local maximum, both

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[1] The second-order conditions are standard. For the special case where rents grow at a constant, positive rate, a sufficient condition for unique maximum (which is interior) is that the elasticity of substitution between capital and land in the production of structure be less than one.

In terms of Figure 1: $\left(\frac{dK}{dT}\right)_{(2)} = -\frac{\Pi_{TT}}{\Pi_{TK}}, \left(\frac{dK}{dT}\right)_{(3)} = -\frac{\Pi_{KT}}{\Pi_{KK}}, \Pi_{TK} = \Pi_{KT} > 0$ by the concavity of $Q(K)$, so that the second-order conditions $\Pi_{TT} < 0, \Pi_{KK} < 0$, and $\Pi_{TT}\Pi_{KK} - \left(\Pi_{KT}\right)^2 > 0$ imply that (2) and (3) are both positively-sloped in T-K space, with (2) having the steeper slope.

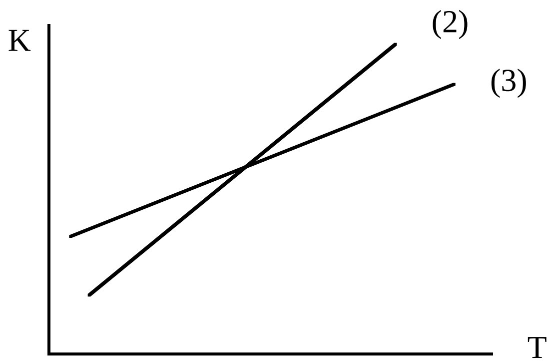

Figure 1: First-order conditions in the Arnott-Lewis model.

(2) and (3) are positively-sloped, and (2) is steeper than (3). Since there will be no ambiguity, K will denote either the variable K or the profit- maximizing value of K; ditto for T.

A site or property tax system is said to be *neutral* if it results in the same (T,K) as solve (2) and (3). This definition is standard. Neutrality implies that the tax system is efficient.

This section examines only site taxation. To proceed with the analysis, additional terminology and notation shall be needed:

n(t)	site rent
τ_n	tax rate on site rent
V(t)	pre-development market value of (vacant) land
P(t)	post-development property value
S(t)	residual site value
τ	tax rate on residual site value under a site value tax system
S(t)	raw site value
τ_S	tax rate on raw site value under a site value tax system

Prior to development, *site rent* equals the market rent on vacant land, which has been assumed equal to zero. Post-development site rent equals property rent minus amortized construction cost. Thus,

$$n(t) = \begin{cases} 0 & t < T \\ r(t)Q(K) - ipK & t > T. \end{cases} \tag{4}$$

Most of the earlier literature employed static models, and hence failed to distinguish between rents and values. The first economists to use dynamic models, the revisionists, employed the residual site value definition of site value. Pre-development residual site value is the pre-development market value of land. Post-development residual site value equals property value minus depreciated structure value. Here the depreciation rate is assumed to be zero. Accordingly,

$$S(t) = \begin{cases} V(t) & t < T \\ P(t) - pK & t > T. \end{cases} \tag{5}$$

As we shall see, with the residual definition of site value, site value taxation is distortionary. It has subsequently been recognized (e.g., Tideman (1982), Netzer (1997), and Ladd (1997) whom will be referred to collectively as *defenders of the orthodoxy*) that the neutrality of site value taxation can be recovered by employing definitions of site value that have the feature that post-development site value is unaffected by the timing and density of development chosen by the market. One such definition of site value is raw site value. Pre-development raw site value — like pre-development residual site value — is the market value of vacant land. Post-development raw site value is what the site would sell for were there no structure on it (even though there in fact is). Thus,

$$\mathsf{S}(t) = \begin{cases} V(t) & t < T \\ \Phi(t) & t > T, \end{cases} \tag{6}$$

where $\Phi(t)$ is "what the site would sell for were there no structure on it", an expression for which shall be derived subsequently.

The literature contains five principal results relating to land/site taxation. The following review states each result, proves it, and provides the economic intuition.

- Result 1: A "pure" land value tax — one which is imposed on the "intrinsic" value of the land, independent of the developer's decision concerning the timing and density of development — is neutral.

Proof: Since the tax payable is independent of the developer's decisions, he views such a tax as a lump sum tax, so it does not affect his decisions. $\square$

This is the idea underlying the neutrality of land value taxation. The neutrality result holds however the intrinsic value of the land is calculated (as long as it is independent of the developer's decisions) and whether the tax rate is constant or variable over time.

- Result 2: A linear, time-invariant tax on site *rent* is neutral.

Proof: The developer chooses T and K to maximize the discounted present value of rent, less construction costs, less tax payments:

$$\max_{T,K} \quad \int_T^\infty r(t)Q(K)e^{-it}dt - pKe^{-iT} - \int_T^\infty \tau_n n(t)e^{-it}dt$$

$$= \int_T^\infty r(t)Q(K)e^{-it}dt - \int_T^\infty ipKe^{-it}dt - \int_T^\infty \tau_n\big(r(t)Q(K) - ipK\big)e^{-it}dt \qquad \text{(using (4))}$$

$$= (1-\tau_n)\int_T^\infty \big(r(t)Q(K) - ipK\big)e^{-it}dt. \qquad (7)$$

The maximizing choices of T and K are independent of τ_n.

In the absence of taxation, the developer chooses T and K to maximize the discounted present value of site rent. With site rent taxation at rate τ_n, the developer chooses T and K to maximize the discounted present value of site rent net of the tax payment. Since the tax equals τ_n times site rent, the maximizing T and K are unaffected by the tax. Site rent is analogous to profit, and the neutrality of site rent taxation analogous to the well-known neutrality of a time-invariant tax on pure profit.

Observe that, with durable structures, a site rent tax whose tax rate is time-varying is not in general neutral. Such a tax does not affect the timing first-order condition, but it does distort the density first-order condition. To see this, consider the top storey of a building in the no-tax situation. Suppose in the early years of the building's life, from T to $\hat{t}$, that the top storey loses money (its net rent is negative: $r(t)Q'(K) - ip < 0$), with these losses being exactly offset in discounted terms by profits in later years. Now impose a site rent tax that is set at a positive rate from T to $\hat{t}$ and at a zero rate thereafter. The top storey is subsidized from T to $\hat{t}$ and incurs no tax liability thereafter. This particular time-varying site rent tax would encourage construction at higher density than in the no-tax situation.

- Result 3: A linear, time-invariant tax on *raw* site value is neutral.

Proof: See Appendix 1.

The intuition for this result was given earlier.

- Result 4: If structures are perfectly malleable or mobile — so that the developer chooses the *function* K(t) — site value taxes are neutral.

Proof: To simplify, assume that $Q'(0) = \infty$, so that development occurs at all points in time. Since capital may be regarded as being mobile — rented at ip per unit per unit time — the market value of land is well-defined. Consequently, there is no ambiguity in the definition of site value, which we denote by $\Sigma(t)$:

$$\Sigma(t) = \max_{K(u)} \int_t^\infty r(u)Q(K(u))e^{-i(u-t)}du - \int_t^\infty ipK(u)e^{-i(u-t)}du - \int_t^\infty \tau_\Sigma(u)\Sigma(u)e^{-i(u-t)}du, \tag{8}$$

when $\tau_\Sigma(u)$ is the site value tax rate at time u. Since structures are perfectly malleable, there is no development timing condition. Differentiating (8) w.r.t. t yields

$$\dot{\Sigma}(t) = -r(t)Q(K(t)) + ipK(t) + i\Sigma(t) + \tau_\Sigma(t)\Sigma(t),$$

and solving gives

$$\Sigma(t) = \max_{K(u)} \int_t^\infty \left(r(u)Q(K(u)) - ipK(u)\right)e^{-\int_t^u (i+\tau_\Sigma(u'))du'}du, \tag{9}$$

from which it is evident that site value taxation does not affect development density. ∎

The intuition is straightforward. Today's site value is essentially independent of today's capital intensity, and future site values completely independent of it. Thus, in deciding on today's capital intensity, the developer views the present value of future site value tax liabilities as a lump sum, and hence his capital intensity decision is unaffected by site value taxation.[2]

Return to the situation where the development decision is completely irreversible — once vacant land is developed at a certain density, it remains at that density forever.

- Result 5: A linear, time-invariant tax on *residual* site value is distortionary

Proof:

[2] It is straightforward to demonstrate that with perfect malleable structures site rent taxation is neutral with a *time-varying* tax rate.

$$S(t) = \begin{cases} \max\limits_{K,T} \int_T^\infty r(u)Q(K)e^{-i(u-t)}du - pKe^{-i(T-t)} - \tau\int_t^\infty S(u)e^{-i(u-t)}dt & t < T \\ \int_t^\infty r(u)Q(K)e^{-i(u-t)}du - pK - \tau\int_t^\infty S(u)e^{-i(u-t)}dt & t > T. \end{cases} \quad (10)$$

Solving S(t) by the now-familiar procedure yields

$$S(t) = \begin{cases} \max\limits_{K,T} \int_T^\infty n(u)e^{-(i+\tau)(u-t)}du & t < T \\ \int_t^\infty n(u)e^{-(i+\tau)(u-t)}du & t > T. \end{cases} \quad (11)$$

The first-order condition with respect to development time is unaffected by the site value tax: $n(T) = 0$. The first-order condition with respect to density is, however, distorted. In particular, the tax on residual site value has the effect of increasing the discount rate on site rent from i to $i + \tau$.

The marginal cost of postponing development equals the rent forgone, and the marginal benefit from postponing development equals the interest on construction costs plus net tax savings. Since residual site value is the same immediately before and immediately after development and since the tax rate on pre-development site value is the same as that on post-development site value, the net tax savings from postponing development equal zero, and the timing first-order condition reduces to what it is in the absence of the residual site value tax.

Two different intuitions for why the residual site value tax distorts the development density first-order conditions are now presented. To simplify, consider the normal case in which rents rise monotonically over time. The developer will add storeys to his building up to the point where the discounted net rent from the top storey equals zero, with the negative net rent in earlier years of the building's life just being offset, in discounted terms, by the positive net rent achieved in later years. The residual site value tax raises the discount rate, which puts greater weight on the earlier years when net rent is negative. Thus, the top storey that just broke even in the absence of the tax loses money when the tax is imposed (holding development time constant), implying that the rise in the discount rate caused by the residual site value tax lowers profit-maximizing development density (holding development time constant). An alternative explanation is as follows. Start with the situation without the residual site value tax. At development time, the top storey of the building just breaks even. In other words, *at* *development time* the increment to residual site value from the top storey is zero. Subsequent to development time, the present value of rent from the top storey increases while the present value of amortized construction costs remains constant. Thus, after development time the increment to residual site value from the top storey is positive (and increasing over time). When, therefore, a residual site value tax is imposed, the top storey adds to the building's discounted tax liability. Imposition of the residual site value tax therefore renders the top storey of the building unprofitable. The residual site value tax therefore discourages density.

The essential difference between raw site value and residual site value taxation should now be apparent. Post-development raw site value is unaffected by the density of development, while in the neighborhood of the optimum post-development residual site value is increasing in the density of development. Thus, imposition of a raw site value tax has no effect on the development density condition, while imposition of a residual site value tax discourages density.

To recapituate: Result 4 was that with perfectly mobile and malleable structure capital, site value taxation is neutral. Results 2, 3, and 5 were derived for the opposite extreme where structure capital is completely immobile and immalleable, but apply as well for intermediate situations where capital can be moved but at a cost and where density can be altered but with adjustment costs. Result 2 was that a linear tax on site rent, at a time-invariant rate, is neutral; result 3 was that a linear tax on raw site value, at a time-invariant rate, is neutral; and result 5 was that a linear tax on residual site value is non-neutral. Thus, the non-neutrality of residual site value taxation derives from a combination of the immobility and immalleability of structure capital, the taxation of site value rather than site rent, and the particular definition of site value employed.

I.2 <u>Practicability of Alternative Definitions of Site Value</u>

The economists who have written recently on site value taxation can be divided into two camps. Defenders of the orthodoxy and modern Georgists[3] view it almost as an article of faith that site value

[3] Henry George was an influential, late-eighteenth century Progressive American reformer who argued in favor of a single tax — a confiscatory tax on land values. Modern Georgists, while not generally adhering to George's view that a confiscatory tax on land values is the *single* tax needed for optimal taxation, subscribe to the view that land value taxation is efficient.

taxation is neutral. They have therefore objected strongly to assertions that site value taxation is distortionary (Tideman (1982), Netzer (1997)). Their view is not unreasonable. Results 1 and 3 of the previous subsection show that land or site value can be defined so that site value taxation is non-distortionary; raw site value is one such definition. The revisionists, however, employ an alternative definition of site value, residual site value. As with raw site value, pre-development residual site value equals the market value of land. In contrast to the definition of raw site value, however, post-development residual site value is measured by property value minus structure value. Under this definition, site value taxation is indeed distortionary, as was shown in Result 5.

The difference between the two camps therefore derives from differences in the definition of post-development site value, which reflects the absence of separate market values for land and structure on a developed site. The choice of the definition of post-development site value should be made on pragmatic grounds. Employing the raw site value definition has the advantage that its use results in a site value tax that is neutral. The residual site value definition has the advantage that its computation, though not without contentious aspects, is relatively straightforward.

Vickrey (1970) characteristically understood the economics of site value taxation before anyone else and also characteristically leaned as far as could reasonably be defended towards the theoretically nice policy: "On the whole --- I am inclined to recommend sticking as closely as possible to a theoretically defined [land or site] value" (p.36). But he also acknowledged that " In the end it seems likely that some degree of departure from the goal of strict neutrality will have to be accepted in order to achieve an acceptable degree of administrative feasibility" (p. 29).

I place more weight on administrative feasibility. The further one moves away from a definition of site value based on market observables, the more capricious[4], unfair, and prone to corruption is site value taxation in practice likely to be. Furthermore, the more capricious the tax system, the greater the amount of wasteful litigation. To reduce appeals, assessments for property tax purposes are now routinely based on hedonic price analysis. A site value tax system that defined site value in a way that

[4] One can imagine a clerk in the Assessment Department of Sticks, Anystate, confronted with the task of computing raw post-development site value!

could not be strongly defended in court, on the basis of market observables, would invite appeals, and for that reason would come to be replaced by a system that defined site value on the basis of market observables. Defining site value as residual site value is not ideal in this regard, since imputed post-development structure value is measured by the depreciated construction costs, which are not simple to estimate. Style obsolescence is hard to measure, and depreciation due to quality deterioration is not only hard to measure but also depends on the level of maintenance chosen which reflects market conditions (Sweeney (1974)). Nevertheless, studies have been undertaken which estimate average rates of depreciation on structures, as captured by age-of-building variables (e.g., Chinloy (1979)), and the results of such studies could be employed to impute post-development structure value. Thus, with hedonic price analysis being employed to estimate post-development property value, residual site value could be imputed using methods that are both straightforward and 'scientific', and therefore readily defensible in court. Hence, on grounds of both administrative costs and fairness, I favor defining site value as residual site value.

Residual site value taxation is, however, distortionary; in particular, it discourages density. A question then arises which, surprisingly, has not been addressed in the literature: Is it possible to design a *property* tax system — defined as linear taxes at different rates that are constant over time on pre-development land value, post-development site value, and post-development structure value — that employs the residual definition of site value and which is neutral or close to neutral? The next section takes up this question.

II. Neutral or Near-neutral Property Taxation

II.1 Analysis and Results

The valuation formulae are now derived for the model of the previous section, but with a property tax system instead of a site value tax system. The property tax system is characterized by a linear tax on pre-development land value at rate τ_V, a linear tax on post-development residual site value at rate τ_S, and a linear tax on post-development structure value at rate τ_K.

Post-development residual site value is

$$S(t) = \int_t^{\infty} r(u)Q(K)e^{-i(u-t)}du - pK - \tau_S \int_t^{\infty} S(u)e^{-i(u-t)}du - \tau_K \int_t^{\infty} pKe^{-i(u-t)}du$$

$$= \int_t^{\infty} \left[r(u)Q(K) - ipK - \tau_S S(u) - \tau_K pK\right]e^{-i(u-t)}du. \qquad (12)$$

Differentiation with respect to t yields

$$\dot{S} = -rQ + (i + \tau_K)pK + (i + \tau_S)S, \qquad (13a)$$

which has the solution

$$S(t) = \int_t^{\infty} \left(r(u)Q(K) - (i + \tau_K)pK\right)e^{-(i+\tau_S)(u-t)}du. \qquad (13b)$$

Pre-development land value, V(t), equals

$$V(t) = \max_{K,T} \left\{S(T)e^{-i(T-t)}\right\} - \tau_V \int_t^{T} V(u)e^{-i(u-t)}du. \qquad (14)$$

Differentiation with respect to t yields

$$\dot{V} = (i + \tau_V)V, \qquad (15)$$

which has the solution (using (13b) and V(T)=S(T) from (14))

$$V(t) = \max_{K,T} \; S(T)e^{-(i+\tau_V)(T-t)}$$

$$= \max_{K,T} \left\{\left[\int_T^{\infty} \left(r(u)Q(K) - (i + \tau_K)pK\right)e^{-(i+\tau_S)(u-T)}du\right]e^{-(i+\tau_V)(T-t)}\right\}. \qquad (15b)$$

The developer chooses T and K so as to maximize the expression in curly brackets in (15b). The first-order conditions are

$$T: \quad \left[-r(T)Q(K) + (i + \tau_K)pK + (\tau_S - \tau_V)V(T)\right]e^{-(i+\tau_V)(T-t)} = 0 \qquad (16)$$

$$K: \quad \left[\int_T^{\infty} \left(r(u)Q'(K) - (i + \tau_K)p\right)e^{-(i+\tau_S)(u-T)}du\right]e^{-(i+\tau_V)(T-t)} = 0\,. \qquad (17)$$

Eq. (16) states that optimal development time occurs when the marginal benefit from postponing development one period equals the marginal cost. The marginal benefit equals the savings from postponing construction cost one period, which equals construction costs times the user cost of capital,

$i+\tau_K$, plus the savings in site value tax payments, $(\tau_S-\tau_V)V(T)$. The marginal cost equals the rent foregone. Eq. (17) states that capital should be added to the site up to the point where the discounted value of the rent attributable to the last unit of capital equals the discounted value of the user cost of the last unit of capital. The post-development residual site value tax has the effect of increasing the post-development discount rate from i to $i+\tau_S$.

The central question to be addressed is whether it is possible to find a neutral property tax system — a set of tax rates ($\tau_V,\ \tau_S,\ \tau_K$) that results in the same T and K as in the absence of taxation and expropriates a specified proportion of value.

Comparing (2) and (16) gives the following condition for neutrality with respect to the development timing condition:

$$\tau_K pK+(\tau_S-\tau_V)V(T)=0, \tag{18a}$$

where V(T) is land value under the property tax. Using (13b) and (14), this becomes

$$\tau_K pK+(\tau_S-\tau_V)\int_T^{\infty}\big(r(u)Q(K)-(i+\tau_K)pK\big)e^{-(i+\tau_S)(u-T)}du=0. \tag{18b}$$

And comparing (3) and (17) gives the following condition for neutrality with respect to the development density decision:

$$\frac{\int_T^{\infty}r(u)e^{-i(u-T)}du}{\int_T^{\infty}r(u)e^{-(i+\tau_S)(u-T)}du}=\frac{i+\tau_S}{i+\tau_K}. \tag{19}$$

<u>Proposition 1</u>: For any functions r(t) and Q(K) and exogenous parameters i and p, there is a property tax system that not only achieves neutrality but also expropriates any specified fraction of land value between 0 and 1.

<u>Proof</u>: Hold K and T at their values at the no-tax optimum. Eq. (19) can be rewritten as $\tau_K=\tau_K(\tau_S)$. Substitution of this function into (18b) yields an equation of the form $\tau_V=\tau_V(\tau_S)$. Thus, for any τ_S, a unique τ_K and τ_V can be determined that result in (18b) and (19) being simultaneously satisfied. In other words, for any value of τ_S, there is a unique property tax system that is neutral.

From (13b), setting $\tau_S = \infty$ expropriates all of the no-tax site value while setting $\tau_S = 0$ expropriates none of the no-tax site value (since with $\tau_S = 0$, τ_K and τ_V are also zero). From (19), $\tau_K(\tau_S)$ is a continuous function of τ_S. From (13b), S(T) is therefore a continuous function of τ_S. Thus, there is a $\tau_S \in [0,\infty)$ such that any specified proportion of site value between 0 and 1 is expropriated.[5] ◡

The general relationship between τ_V, τ_S, and τ_K in a neutral tax system is complex and is investigated in Appendix 2. A complete characterization of the relationship between τ_V, τ_S, and τ_K for the special case where rents grow exponentially over time can, however, easily be obtained. The growth rate of rents is denoted by $\eta(> 0)$.[6]

Proposition 2: When rents grow at a constant rate η, a neutral property tax system has the properties that:

i) $\tau_K = \tau_S\left(-\frac{\eta}{i + \tau_S - \eta}\right)$

ii) $\tau_V = 0$

Proof: i) follows directly from (19). ii) then follows from (18b), after substitution of i). ◡

We provide two different intuitive explanations for the results in Proposition 2. The first is casual, the second exact. The residual site value tax system considered in the previous section is a special case of the class of property tax systems considered here, with $\tau = \tau_V = \tau_S$ and $\tau_K = 0$. Recall (proof of Result 5) that that tax system had no effect on the development timing condition but caused the development density condition to change in such a way that discourages density. Take the residual site value tax system as the starting point and consider how τ_S, τ_V, and τ_K should be modified to restore neutrality. First, capital should be subsidized to offset the depressing effect of residual site value taxation on development density. But from (16a), the subsidization of capital reduces the marginal benefit from postponing development. The development timing condition, which was undistorted with residual site

[5] Observe that the proof generalizes to the situation where i, p, and Q(K) are all functions of time.

value taxation, becomes distorted, leading to excessively early development. This can be corrected by setting the pre-development land value tax rate below the post-development site value tax rate. This intuition suggests that, a neutral property tax system which raises positive revenue has $\tau_S > \tau_V$ and $\tau_K < 0$. This intuition is consistent with Proposition 2 which concerns a special case, but is not correct in general (see Appendix 2).

The precise intuition is based on a result that is sufficiently important that we present it as:

Proposition 3: When rents grow at the constant rate η, the neutral property tax system described in Proposition 2 is equivalent to a site rent tax system with the constant tax rate $\tau_n = \dfrac{\tau_S}{i + \tau_S - \eta}$.

Proof: Since both tax systems are neutral and hence have the same development time and density, it suffices to demonstrate that the time paths of tax revenue collected under the two tax systems coincide. For both, the tax revenue collected prior to development is zero. After development, the time path of revenue collected under the site rent tax is

$$R(t) = \tau_n \big(r(t)Q(K) - ipK \big). \tag{20}$$

With a property tax system, the time path of revenue collected is

$$\begin{aligned} R(t) &= \tau_S S(t) + \tau_K pK \\ &= \tau_S \frac{r(t)Q(K)}{i + \tau_S - \eta} - \frac{\tau_S (i + \tau_K) pK}{i + \tau_S} + \tau_K pK \quad \text{(using (13b))}. \end{aligned} \tag{21}$$

Now substitute property i) of the neutral property tax system into (21):

$$R(t) = \frac{\tau_S}{i + \tau_S - \eta} \quad \big(r(t)Q(K) - ipK \big). \tag{22}$$

With $\tau_n = \dfrac{\tau_S}{i + \tau_S - \eta}$, the two tax revenue streams are identical. ˛

Proposition 3 has an immediate:

Corollary: With rents growing at a constant rate η: i) at every point in time, the ratio of property tax revenue collected under a neutral property tax system to site rent equals $\tau_S / (i + \tau_S - \eta)$; and

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[6] A necessary condition for a local maximum is that rents be growing at development time.

ii) the ratio of the present value of property tax revenue collected under a neutral property tax system to the no-tax pre-development land value (for $t < T$) or no-tax post-development residual site value (for $t > T$) — which is one measure of the proportion of value expropriated by the tax system — equals $\tau_S/(i + \tau_S - \eta)$.

Part i) of the Corollary follows immediately from the proof to Proposition 3. Part ii) follows from part i). The Corollary is related to the second part of Proposition 1. There it was shown that there exists a neutral property tax system that expropriates any desired proportion of the no-tax site value. Proposition 3 gives the exact relation between τ_S and the proportion of no-tax site value expropriated, for the special case of a constant growth rate of rents.

Another useful result for the situation where rents grow at a constant rate is given in:

Proposition 4: Under the neutral property tax system described in Proposition 2, property value at development time, site value at development time, and structure value are in the proportions: $\tau_S - \tau_K$, $-\tau_K$, τ_S.

The rest of this section discusses the implications of the above results for the design of a practicable and neutral or close-to-neutral property tax system.

II. 2 Discussion

Since site rent taxation at a time-invariant rate is neutral, why not employ such a site rent tax rather than a more complex neutral property tax system? The primary reason is presumably that site rents are generally unobservable and would be difficult to estimate. While the estimation of pre-development land value, post-development property value, and structure value is by no means trivial, there is a wealth of practical experience to draw on. This observation points to the importance of considering the informational feasibility of implementing alternative property tax systems.

Is implementation of the neutral property tax system derived in the previous subsection informationally feasible? At first glance, the answer would appear negative since the optimal tax rates on a parcel depend on that parcel's future time path of rents, which is of course unknown. While the market does not directly signal expectations concerning future rents, it does provide some information relevant to computing the optimal tax rates: prior to development, the market value of the parcel, and immediately after development, development time, development density, and property value. The question is whether this information is sufficient to calculate the set of tax rates that achieves neutrality and raises the desired revenue or expropriates the desired proportion of value. An incomplete answer, in terms of the model (which entails the assumptions, *inter alia*, that the rent on vacant land is zero and that the interest rate is time-invariant,) is that this information is sufficient if post-development rents grow at a constant rate, but not generally otherwise. Under these circumstances, Proposition 2 indicates that $\tau_V = 0$, Proposition 4 that $\tau_K = -\tau_S\left(S(T)/pK\right)$, and $(A2.9'')$ in Appendix 2 that $\tau_S = i\varepsilon pK\big/\left((1-\varepsilon)pK + S(T)\right)$, where ε is the desired proportion of value to expropriate. This assumes that developers take tax rates as parametric. If, however, the government were to compute τ_K for a particular property from $\tau_K = -\tau_S\left(S(T)/pK\right)$ on the basis of that property's $S(T)/pK$, the developer would take into account that by altering the density of development he would alter the subsidy rate on capital, which would lead to non-neutrality.[7] This problem is easily overcome by setting a particular property's τ_K on the basis of the $S(T)/pK$ ratio for "comparable", recently developed properties.

When, however, rents do not grow at a constant rate, the situation is more complicated. Neutrality in general requires a non-zero tax rate on (pre-development) land value. But prior to development, the only information the market provides relevant to the future rental stream is land value.

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[7] Substitution of $\tau_K = -\tau_S\left(S(T)/pK\right)$ into (13b) gives

$$= \int_T^\infty \left(r(u)Q(K) - ipK + \tau_S S(T)\right)e^{-(i+\tau_S)(u-T)}du$$

$$\Rightarrow S(T) = \frac{i+\tau_S}{i}\int_T^\infty \left(r(u)Q(K) - ipK\right)e^{-(i+\tau_S)(u-T)}du. \qquad \text{(i)}$$

Thus

$$V(t) = \max_{K,T} \left\{\left(\frac{i+\tau_S}{i}\int_T^\infty \left(r(u)Q(K) - ipK\right)e^{-(i+\tau_S)(u-T)}du\right)e^{-(i+\tau_V)(T-t)}\right\}, \qquad \text{(ii)}$$

which yields the following first-order conditions for T and K:

$$T: \frac{i+\tau_S}{i}e^{-(i+\tau_V)(T-t)}\left[-r(T)Q(K) + ipK + \left(\tau_S - \tau_V\right)V(T)\right] = 0 \qquad \text{(iii)}$$

$$K: \frac{i+\tau_S}{i}e^{-(i+\tau_V)(T-t)}\left[\int_T^\infty \left(r(u)Q'(K) - ip\right)e^{-(i+\tau_S)(u-T)}du\right] = 0. \qquad \text{(iv)}$$

Timing neutrality requires $\tau_S = \tau_V$. But with $\tau_S = \tau_V > 0$ the density decision is distorted — density is discouraged.

This and the time at which the land value tax is first imposed do not provide enough information to compute τ_V.

Intuition suggests that practically the market is sufficiently uncertain concerning the time path of future rents that it has only weak beliefs concerning how the time path of future rents will differ from a constant-growth-rate time path. This suggests that employing a property tax that would be neutral if the future growth rate of rents were constant would normally come close to achieving neutrality. Examination of this conjecture will require extending the model to allow for uncertainty (which could build on Capozza and Li ( )).

The above argument suggests that design of a near-neutral property tax system is informationally feasible. What of administrative feasibility? The analysis of the previous subsection applied to an isolated property. If the tax system were to be applied as modeled, every property would have its own post-development site value tax rate and structure value subsidy rate, which would be very cumbersome. And since all tax rates on developed properties would have been set in the past, the government would have no discretion to raise or lower tax revenues in the short run. Clearly, administrative feasibility requires adapting the property tax system analyzed in the previous subsection.

The objective therefore is to find a property tax system that is informationally and administratively feasible and that comes close to being neutral. It was argued earlier that a residual site value tax system is both informationally and administratively feasible. Its weakness is that it discourages density. These observations suggest the following simple adaptation of the neutral property tax system analyzed earlier. After development, impose a residual site value tax along with a structure investment tax credit. The tax rate on pre-development land value would be zero; the tax rate on post-development site value would be set annually according to the government's revenue requirements, etc.; and the tax credit rate on structure investment would be set annually with the objective of achieving neutrality with respect to the timing and density of development.

Implementation of such a tax system would require addressing a number of practical issues: How finely should the tax credit rate on structure investment be varied over space? How should the transition from the current system to this system be designed so as to achieve a smooth revenue stream, to avoid causing a building boom or bust, and to be politically acceptable, which requires among other things not generating substantial capital losses on any major class of properties? And how should the tax credit rate on structure investment be determined? This tax system is only one of many that might attain the best balance between practicability and deviation from neutrality, and has been presented more as a basis for discussion than as a proposal.

The derivation of neutral property tax systems presented in the previous subsection made a large number of simplifying assumptions. Future research should investigate how the results need to be modified when account is taken of time variation in the interest rate and the price of structure capital, technological change in construction, maintenance and depreciation, the possibility of redevelopment, and uncertainty, and when time variation of the tax rates is considered.

III. The Deadweight Loss from Non-Neutral Property and Site Value Taxation[8]

Remarkably, it appears that no one has investigated the deadweight loss associated with alternative property tax systems from the modern perspective which correctly views the tax as a tax on value (rather than rent) in a dynamic economy. The aim of the section is to present some preliminary results rather than an exhaustive analysis.

III.1 $\tau_V > 0$ from time immemorial

The first case considered is that where a positive tax on land value has been applied from $t = -\infty$. To simplify the algebra slightly, values are evaluated at t=0. Where R(t) is the value of tax revenue discounted or brought forward to time t,

$$R(0) = \tau_V \int_{-\infty}^{T} V(t) e^{-it} dt + \tau_K \int_{T}^{\infty} pKe^{-it} dt + \tau_S \int_{T}^{\infty} S(t) e^{-it} dt. \tag{25}$$

Also,

$$S(T) = \int_{T}^{\infty} r(t) Q(K) e^{-i(t-T)} dt - pK - \tau_K \int_{T}^{\infty} pKe^{-i(t-T)} dt - \tau_S \int_{T}^{\infty} S(t) e^{-i(t-T)} dt. \tag{26}$$

Substituting (26) into (25), and simplifying, yields

$$\mathsf{R}(0) = \tau_V \int_{-\infty}^{T} V(t)e^{-it}dt - pKe^{-iT} - S(T)e^{-iT} + \int_{T}^{\infty} r(t)Q(K)e^{-it}dt. \tag{27}$$

Finally, substituting

$$V(t) = V(T)e^{-(i+\tau_V)(T-t)} \tag{28}$$

into (27), and using $S(T) = V(T)$, gives

$$\mathsf{R}(0) = \int_{T}^{\infty} r(t)Q(K)e^{-it}dt - pKe^{-iT}. \tag{29}$$

To evaluate the deadweight loss from the tax system, additional notation shall be employed. Let ^{b} denote the pre-tax situation, ^{a} the after-situation, $\mathsf{L}(t)$ the landowner's surplus evaluated at t, and $\mathsf{D}(t)$ deadweight loss at t.

The landowner's surplus evaluated at t=0 is

$$\mathsf{L}^b(0) = \left(V(T)e^{-iT}\right)^b \tag{30a}$$

$$\mathsf{L}^a(0) = \left[\left(V(T) - \tau_V \int_{-\infty}^{T} V(t)e^{i(T-t)}dt\right)e^{-iT}\right]^a = 0 \qquad \text{(using (27) and (29)).} \tag{30b}$$

Thus, whatever the tax rate, as long as $\tau_V > 0$, the landowner's surplus is driven to zero by the tax system.

Since, in the partial equilibrium model employed in the paper, the structure rent stream and hence the utility of renters is unaffected by taxation, the loss in social surplus is simply the sum of the loss in landowner and government surplus. Thus,

$$\mathsf{D}(0) = \mathsf{L}^b(0) - \left(\mathsf{R}(0) + \mathsf{L}^a(0)\right)$$

$$= \left(V(T)e^{iT}\right)^b - \mathsf{R}(0). \tag{31}$$

Defining

$$Y^b \equiv \left(\int_{T}^{\infty} r(t)Q(K)e^{-it}dt - pKe^{-iT}\right)^b$$

and Y^a accordingly,

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[8] I would like to thank seminar participants in the Urban Land Division, Faculty of Commerce, University of British Columbia, for suggesting I treat the topic of this section.

$$\mathsf{D}(0) = Y^b - Y^a. \tag{32}$$

Note from (29) and (30b) that whatever the tax rate on land value, the government expropriates the full social surplus. If the tax rate on land value is infinitesimal and positive and the tax rates on residual site value and structure value infinitesimal or zero, the property tax system is effectively neutral. These two observations together lead to

<u>Proposition 5</u>: When a positive tax on land value is applied from time immemorial:

i) the government expropriates the entire social surplus from the land in the form of tax revenues;

ii) if the tax rate on land is infinitesimal, and the tax rates on residual site value and structure value zero are infinitesimal, then the property tax system is essentially neutral; thus, with such a property tax system, the government is able to expropriate the entire surplus with essentially no distortion.

This is a striking result. At the same time, it is obvious that property tax systems have not been in place since time immemorial. Thus, the result is of theoretical interest or an illuminating extreme case.

III.2 <u>Other property tax systems</u>

i) $\tau_V = 0$

Proceeding as for the previous case,

$$\mathsf{R}(0) = \left(\int_T^\infty r(t)Q(K)e^{-it}dt - pKe^{-iT} - S(T)e^{-iT}\right)^a, \tag{33}$$

$\mathsf{L}^b(0) = \left(V(T)e^{-iT}\right)^b$ and $\mathsf{L}^a(0) = \left(V(T)e^{-iT}\right)^a$. Thus,

$$\mathsf{D}(0) = \left(V(T)e^{-iT}\right)^b - \mathsf{R}(0) - \left(V(T)e^{-iT}\right)^a$$

$$= Y^b - Y^a \qquad (\text{since } V(T) = S(T)).$$

ii) $\tau_V > 0$ and the land value tax first applied at t=I.

In this case,[9]

[9] $\left(\tau_V \int_I^T V(t)e^{-it}dt\right)^a = \left(\tau_V \int_I^T V(T)e^{-(i+\tau_V)(T-t)}e^{-it}dt\right)^a$

$= \left(V(T)e^{-iT}\left(1 - e^{-\tau_V(T-I)}\right)\right)^a$

$$\mathsf{R}(0) = \left(\tau_V \int_I^T V(t)e^{-it}dt - pKe^{-iT} - S(T)e^{-iT} + \int_T^\infty r(t)Q(K)e^{-it}dt\right)^a$$

$$= \left(\int_T^\infty r(t)Q(K)e^{-it}dt - pKe^{-iT} - S(T)e^{-iT}e^{-\tau_V(T-I)}\right)^a, \qquad (34)$$

$$\mathsf{L}^b(0) = \left(V(T)e^{-iT}\right)^b \qquad (35a)$$

$$\mathsf{L}^a(0) = \left[\left(V(T) - \tau_V \int_I^T V^a(t)e^{i(T-t)}dt\right)e^{-iT}\right]^a$$

$$= \left(V(T)e^{-iT}e^{-\tau_V(T-I)}\right)^a \qquad (35b)$$

Thus, from (34) and (35b)

$$\mathsf{D}(0) = \left(V(T)e^{-iT}\right)^b - \mathsf{R}(0) - \left(V(T)e^{-i(T)}e^{-\tau_V(T-I)}\right)^a$$
$$= Y^b - Y^a.$$

The above results are drawn together in

<u>Proposition 6</u>: With $\tau_V \geq 0$:

a) Evaluated at t=0, the deadweight loss due to the property tax is

$$\mathsf{D}(0) = \left(\int_T^\infty r(t)Q(K)e^{-it}dt - pKe^{-iT}\right)^b - \left(\int_T^\infty r(t)Q(K)e^{-it}dt - pKe^{-iT}\right)^a.$$

b) Evaluated to t=0, the revenue collected from the property tax is

$$\mathsf{R}(0) = \begin{cases} \left(\int_T^\infty r(t)Q(K)e^{-it}dt - pKe^{-iT}\right)^a & \tau_V > 0 \text{ from time immemorial} \\ \left(\int_T^\infty r(t)Q(K)e^{-it}dt - pKe^{-iT} - S(T)e^{-iT}\right)^a & \tau_V = 0 \\ \left(\int_T^\infty r(t)Q(K)e^{-it}dt - pKe^{-iT} - S(T)e^{-iT}e^{-\tau_V(T-I)}\right)^a & \tau_V > 0 \text{ from } t = I \end{cases}$$

The first part of the Proposition accords with intuition. $\int_T^\infty r(t)Q(K)e^{-it}dt - pKe^{-iT}$ is the social surplus generated from the land, conditional on development at time T and density K, discounted or brought forward to t=0. Thus, $\mathsf{D}(0)$ is the loss in the discounted social surplus from the land due to the property tax distorting the choices of T and K.

The second part of the Proposition points out conceptual difficulties in comparing the efficiency of alternative property tax systems. The natural way to compare the efficiency of two tax systems is to hold the revenue raised constant and to compare the deadweight losses of the two systems. But in comparing two tax systems with $\tau_V > 0$, or in comparing one tax system with $\tau_V > 0$ with another with $\tau_V = 0$, the comparison depends on when the land value tax is first applied — what the value of I is. Another difficulty is that when $\tau_V > 0$, the deadweight loss is not minimized when the tax revenue collected is zero. This shows up most starkly when $\tau_V > 0$ from time immemorial. Suppose that the tax rate on land value is infinitesimal and positive, while the tax rates on post-development residual site value and post-development structure value are zero or infinitesimal. As recorded in Proposition 5, such a tax system is not only neutral but also expropriates the full value of the land: $R(0) = Y^b$.

In the next subsection these difficulties are sidestepped by comparing property tax systems with $\tau_V = 0$. But then the subsequent subsection will confront these difficulties.

III. <u>The deadweight loss from the common property tax $\tau_V = 0$, $\tau_S = \tau_K = \tau$</u>

The common property tax system taxes pre-development land value on the basis of what the value of the land would be if it were held in agricultural use forever, A(t) — *agricultural land value*. Since agricultural rents are zero, A(t)=0, so that the effective tax rate on pre-development land value is $\tau_V = 0$. The common property tax system also effectively applies the same tax rate to post-development residual site value and post-development structure value: $\tau_S = \tau_K = \tau$.

To simplify the algebra, only the situation where rents grow at a constant rate shall be considered. Part a) obtains some general results, while part b) investigates a numerical example.

a) some general results

Elementary algebra yields

$$V^a(0) = \left[\frac{r(T)Q(K)}{i+\tau-\eta} - pK\right]e^{-iT}$$

$$= \left[\frac{r(0)e^{\eta T}Q(K)}{i+\tau-\eta} - pK\right]e^{-iT}. \tag{36a}$$

The corresponding first-order conditions for T and K are:

$$\text{T:} \qquad -r(T)Q(K)\left(\frac{i-\eta}{i+\tau-\eta}\right) + ipK = 0 \tag{36b}$$

$$\text{K:} \qquad -r(T)Q'(K)\left(\frac{1}{i+\tau-\eta}\right) - p = 0. \tag{36c}$$

(36b) and (36c) together imply

$$\frac{Q(K)}{Q'(K)K} = \frac{i}{i-\eta}. \tag{36d}$$

This well-known result from Arnott and Lewis (1979) — that with constant growth rate of rents, the common property tax has no effect on development density — is shown diagrammatically in Figure 2A. The result will considerably simplify the analysis.

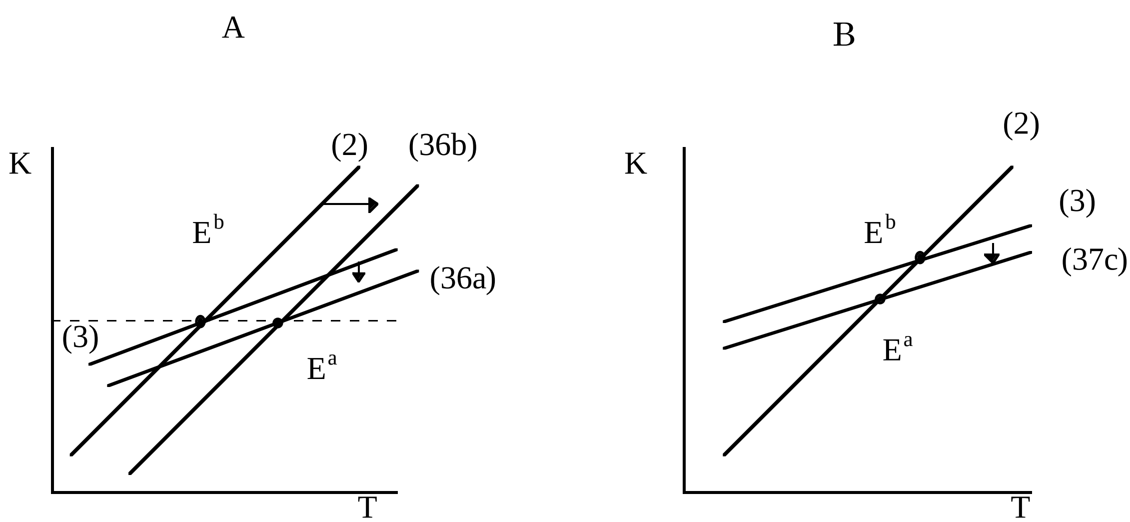

Figure 2: A: Effect of the common property tax
B: Effect of a residual site value tax

The results given in Proposition 6, combined with those given in (36a)-(36d), imply

$$D(0) = V^b(0)\left(1 - \frac{i\tau + i\eta - \eta^2}{(i-\eta)\eta}\left(\frac{i-\eta}{i+\tau-\eta}\right)^{i/\eta}\right) \tag{36e}$$

$$R(0) = V^b(0)\left(\frac{i\tau}{\eta(i-\eta)}\left(\frac{i-\eta}{i+\tau-\eta}\right)^{i/\eta}\right) \tag{36f}$$

$$V^b(0) = \frac{pK\eta}{i-\eta}\left(\frac{Q(K)r(0)}{ipK}\right)^{i/\eta} \quad (36g)$$

$$T^b = \frac{1}{\eta}\ell n\left(\frac{ipK}{Q(K)r(0)}\right) \quad (36h)$$

$$T^a = T^b + \frac{1}{\eta}\ell n\left(\frac{i+\tau-\eta}{i-\eta}\right) \quad (36i)$$

Define marginal deadweight loss (MDWL) to be the efficiency loss generated by the marginal dollar of tax revenue collected. It is straightforward to calculate from (36e) and (36f) that[10]

$$MDWL = \tau/(\eta-\tau). \quad (36j)$$

Other values of interest can be calculated straightforwardly from the equations. Note, in accord with Arnott-Lewis and Figure 2A, that a rise in the tax rate causes development to be postponed.

Two interesting results are given in

Proposition 7: With the common property tax ($\tau_V = 0$, $\tau_S = \tau_K = \tau$) and a constant rental growth rate, the revenue-maximizing tax rate equals the growth rate of rents, and the marginal deadweight loss from the tax is $\tau/(\eta-\tau)$.

The proof of the first result follows directly from (36f). The result suggests that some common property tax systems may be "on the wrong side of the Laffer curve".

b) numerical example

Choose units such that $K^b = Q^b = p = 1$, and suppose that[11] $r(0) = i = .03$ and $\eta = .02$. The results for several tax rates are recorded in Table 1.

τ	T	K	Q	L(0)=V(0)	D(0)	R(0)
0	0	1	1	2	0	0
.01	34.657	1	1	.70711	.23223	1.06066

[10] When $\tau > \eta$, MDWL is negative, since a rise in the tax rate causes deadweight loss to rise and tax revenue to fall.

[11] $r(0)$ is chosen so that in the no-tax situation the land is developed at t=0.

.02	54.931	1	1	.38490	.46040	1.15470
.03	69.315	1	1	.25000	.62500	1.12500

Table 1: Numerical example with common property tax

Because, with the common property tax, profit-maximizing structural density is independent of the property tax rate, the above results hold independent of the form of the structure production function. The results indicate that the effects of the common property tax are substantial. With i=.03 and $\eta = .02$, a two-percent tax rate causes land value to fall to only 19% of pre-tax value, generates a deadweight loss of 23% of pre-tax value, and causes development of the land to be postponed 55 years!

These are the effects of imposing the property tax on a single parcel of land. The general equilibrium effects would be considerably more complicated, and should be analyzed in the context of a general equilibrium model, such as a growing, fully-closed monocentric city.

III.4 The deadweight loss from residual site value taxation ($\tau_K = 0$, $\tau_V = \tau_S = \tau$)

a) general results

This case is more difficult since the results depend on at what time, I, the site value tax was initially imposed, and since the tax affects both the timing and density of development.

Elementary algebra yields

$$V^a(0) = \left[\frac{r(T)Q(K)}{i+\tau-\eta} - \frac{ipK}{i+\tau}\right]e^{-(i+\tau)T}$$

$$= \left[\frac{r(0)e^{\eta T}Q(K)}{i+\tau-\eta} - \frac{ipK}{i+\tau}\right]e^{-(i+\tau)T}. \quad (37a)$$

The corresponding first-order conditions for T and K are

T: $-r(T)Q(K) + ipK = 0$ (37b)

K: $\frac{r(T)Q'(K)}{i+\tau-\eta} - \frac{ip}{i+\tau} = 0$ (37c)

Comparing (37b) and (37c) yields

$$\frac{Q(K)}{Q'(K)K} = \frac{i+\tau}{i+\tau-\eta}. \tag{37d}$$

The rise in τ has no effect on the timing first-order condition, but in T-K space causes the development density condition to shift down; the result is earlier development at lower density (Figure 2B).

The sensitivity of development density to τ depends on the elasticity of substitution between capital and land in the production of structure. To obtain results that are both interesting and analytically tractable, it is therefore reasonable to assume that the structure production has the CES form:

$$Q(K) = c_0\left(1+c_1K^{\rho}\right)^{1/\rho},\ \sigma = \frac{1}{1-\rho},$$

where $c_0,\ c_1 > 0$ and $\rho \in (-\infty,\ 1)$ are exogenous parameters and σ is the elasticity of substitution. (As noted in footnote 1, a necessary condition for a local optimum is that $\sigma < 1$.) With this particularization of the structure production function, (37d) becomes

$$K = \left(\frac{\eta c_1}{i+\tau-\eta}\right)^{\sigma/(1-\sigma)}.$$

To further simplify the algebra, assume that $\sigma = 1/2\left(\rho = -1\right)$, which is in fact near the midpoint of empirical estimates (McDonald (1981)), so that $Q(K) = c_0\left(\frac{K}{c_1+K}\right)$. Solving (37b) and (37d) for T and K for this class of structure production functions yields

$$T^b = \frac{1}{\eta}\ell n\left(\frac{i^2pc_1}{(1-\eta)c_0r(0)}\right) \qquad T^a = T^b - \frac{1}{\eta}\ell n\left[\left(\frac{i+\tau-\eta}{i+\tau}\right)\left(\frac{i}{i-\eta}\right)\right] \tag{38a,b}$$

$$K^b = \frac{\eta c_1}{i-\eta} \qquad K^a = \frac{\eta c_1}{i+\tau-\eta} \tag{38c,d}$$

Also,

$$V^b(0) = \frac{pK^b\eta}{i-\eta}\left(\frac{(i-\eta)c_0r(0)}{i^2pc_1}\right)^{i/\eta} \tag{38e}$$

$$D(0) = V^b(0)\left(1-\left(\frac{i+\tau+\eta}{i-\eta}\right)^{(i-\eta)/\eta}\left(\frac{i}{i+\tau}\right)^{i/\eta}\right) \tag{38f}$$

$$R(0) = V^b(0)\left(\frac{i+\tau-\eta}{i-\eta}\right)^{(i-\eta)/\eta}\left(\frac{i}{i+\tau}\right)^{i/\eta}\left(1-\frac{i}{i+\tau}e^{-\tau\left(T^a-I\right)}\right). \tag{38g}$$

The difficulty in analyzing the residual site land tax is now evident. The revenue generated by the tax depends on when it was first applied; so too do the increase in revenue from a unit increase in the tax rate, the marginal deadweight loss, and the revenue-maximizing tax rate.[12] But there is no natural choice of I. The same difficulty arises in comparing the residual site value tax (or any tax for which $\tau_V > 0$) with any other type of property tax.

b) numerical example

Recall that in the numerical example of the previous subsection, the no-tax development time is t=0. To allow comparison with that example, let $I = -50$. Thus, the comparison entails looking at a parcel of land that in the absence of taxation would be developed today, and comparing the choice fifty years ago between a residual site value tax system and a common property tax system. Alternatively, one may interpret t=0 to be fifty years into the future, in which case the comparison entails the choice today between applying the two tax systems to a parcel that in the absence of taxation would be developed fifty years hence.

To maintain comparability with the example of the previous subsection, the following parameter values are assumed: $c_1 = .5$, $c_0 = 1.5$ (these two parameters imply that $K^b = 1$ and $Q^b = 1$, which accords with the common property tax example), $r(0) = .03$, $i = .03$, $\eta = .02$, and $p = 1$. The results are recorded in Table 2

τ	T	K	Q	L(0)	D(0)	R(0)
0	0	1	1	2(=V(0))	0	0
.01	-20.273	.5	.75	1.02350	.16289	.81361
.02	-29.390	$.\dot{3}$	.6	.63966	.39003	.97031
.03	-34.657	.25	.5	.44626	.58579	.96795

[12] Recall, however, (Proposition 5) that with $I = -\infty$, the revenue-maximizing tax rate is infinitesimally positive. The tax expropriates all of the social surplus which is maximized under a neutral tax, and the infinitesimally positive tax is essentially neutral.

Table 2: Numerical example with residual site value tax

Comparing the two tables suggests that, with the assumed parameter values, the common property tax is more efficient than the residual site value tax, first because the maximum revenue that can be raised under the common property tax is higher, and second because the common property tax at rate .01 raises more revenue and generates less deadweight loss than a residual site value tax at rate .02. We know, however, that with $I = -\infty$ and $\tau = 0^+$ the residual site value tax expropriates the maximum possible social surplus (and therefore with no deadweight loss). This implies that there is some value of I below which residual site value taxation dominates common property taxation. Also, when the elasticity of substitution between land and capital in the production of housing is zero, residual site value taxation generates no distortion. Thus, which is more efficient, the common property tax or a residual site value tax, depends on parameter values.

The above analysis and examples have indicated the conceptual difficulties of comparing one property tax system having $\tau_V > 0$ with another having $\tau_V = 0$. Similar difficulties arise in comparing two property tax systems when both have $\tau_V > 0$. The difficulties disappear, however, in comparing two property tax systems which exempt pre-development land value. Thus, the analysis presented in this section could be extended straightforwardly to examine the efficiency gains from switching from a common property tax ($\tau_V = 0$, $\tau_K = \tau_S$) to one which taxes structure value at a lower rate than residual site value ($\tau_V = 0$, $\tau_K < \tau_S$).

IV. Conclusion

The paper started by providing a synthetic overview of the literature on site/land taxation. The orthodox view is that the taxation of land is non-distortionary, whether it be land rent or land value. The basic idea is that the value of land or the rent it commands is independent of decisions concerning its use by the current owner and/or tenant. If that is the case, taxation of such value or rent is then regarded by the agent who decides on the land's use as a lump-sum tax, and does not therefore affect his decisions concerning its use. No contributor to the modern, mainstream literature on the subject disputes this view. The disagreement instead centers on how land should be valued for property tax purposes *after* it has been developed — when there is a durable and immobile structure on the site. Since there is no market for such land, its value is not logically determinable. There are two broad points of view concerning how the value developed land should be imputed for property tax purposes. Defenders of the orthodoxy argue that land value should be defined in such a way that its taxation is neutral. There are many ways this can be done. One such definition was treated, *raw site value* — what the site would sell for *if it were undeveloped*, even after it has in fact been developed. The problem with using this definition is that post-development raw site value would be so difficult to estimate that the resulting tax system would be inequitable, capricious, and subject to abuse. The revisionists have employed an alternative definition of land value for a built-on site: property value minus structure value, which was termed *residual site value*. Property value can be estimated using current assessment practice based on hedonic analysis, while structure value can be estimated by applying an estimated depreciation rate to original construction costs. Estimating residual site value would therefore be relatively easy. However, using residual site value as a basis for taxation violates neutrality; in particular, holding fixed development time, it discourages density.

Reasonable men may differ concerning which of the two broad approaches to site value taxation is preferable. I came down on the side of residual site value taxation. Vickrey, whose logic is always impeccable, favored a definition which comes as close as is administratively feasible to preserving neutrality.

This paper contributed to the revisionist literature. The revisionists have demonstrated that residual site value tax is distortionary, but have not taken the next step of asking the question: Is it possible to design a *property* tax system employing the residual definition of site value on built-on land that is neutral? That was the central question addressed in this paper.

A property tax system was defined as a triple of linear taxes: a tax on pre-development land value at rate τ_V, a tax on post-development residual site value at rate τ_S, and a tax on post-development structure value at rate τ_K. To address the question, a partial equilibrium model was employed which looked at a single developable site. The simplifying assumptions were made that once a site is developed at a particular density it remains at that density forever, and that the rent on undeveloped land is zero. The main result was that for this model there is indeed a combination of the three tax rates that raises a given level of discounted tax revenues and achieves neutrality. The basic intuition is simple. The government has three objectives — not distorting the development timing decision, not distorting the development density condition, and extracting a pre-determined proportion of value — and three instruments to achieve these objectives. This intuition suggests that the neutrality result extends to considerably more realistic models than the one employed in this paper.

The paper then calculated the three tax rates that achieve neutrality for the special case in which the rental growth rate is constant over time. The tax rate on pre-development land value should be zero, the tax rate of post-development residual site value should be set so as to achieve the desired expropriation of value, and the structure value tax rate should be negative. One intuition is that, under the assumptions made, this property tax system is equivalent to a tax on net site rent at a time-invariant rate, which was earlier shown to be neutral.

The paper then briefly discussed the relevance of the theoretical results for the design of practical property tax systems, taking into account considerations of informational and administrative feasibility, and of political acceptability. Two general points were made. The first was that the model requires considerable elaboration, and that the neutral property tax system generated by any theoretical model would need to be extensively adapted for practical application. The second was that, these cautions notwithstanding, it should be possible to design a practicable property tax system that comes close to achieving neutrality. A sample system was put forward as a basis for discussion: Tax exemption for land prior to development; residual site value taxation after development, with a structure investment tax credit being used to offset the depressing effect of residual site value taxation on development density.

The paper went on to derive the deadweight loss associated with non-neutral site value and property value tax systems. Two particularly noteworthy results were obtained. The first was that a tax on pre-development land value at an infinitesimal rate, applied from infinitely far in the past up to development time, is not only neutral but achieves full expropriation of value. The second was that, when structure rents grow at a constant rate, the common property tax system — which (with zero agricultural rent) applies a zero tax rate to pre-development land value and equal tax rates to post-development residual site and structure values — has a revenue-maximizing tax rate equal to the growth rate of rents.

One final remark. The literature on property taxation, to which the paper has contributed, has evolved largely independently of other important developments in public economics. There is an extensive literature on neutral capital taxation (e.g., Samuelson (1964), King and Fullerton (1984)). The two literatures should be integrated, not only to develop results on neutral capital c property taxation, but also to investigate second-best efficient property taxation when capital taxation is distorted, and *vice versa*. There is also an extensive literature on the design of optimal tax systems, which takes into account the equity-efficiency tradeoffs produced by asymmetries in information. It is time for the property tax to be considered as one component of a broad tax system rather than being examined in isolation.

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Appendix 1

Proof of Result 3:

Prior to development, raw site value equals the value of the vacant land, $V(t)$. After development, raw site value equals the value of the land were it still undeveloped.

Using (6), the value of vacant land for t<T is

$$V(t) = \max_{K,T} \left[\int_T^\infty r(u)Q(K)e^{-i(u-t)}du - pKe^{-i(T-t)} - \tau_S \int_t^\infty S(u)e^{-i(u-t)}du\right]$$

$$= \max_{K,T} \left[\int_T^\infty r(u)Q(K)e^{-i(u-t)}du - pKe^{-i(\tau-t)} - \tau_S \int_t^T V(u)e^{-i(u-t)}du - \tau_S \int_T^\infty \Phi(u)e^{-i(u-t)}du\right]. \quad \text{(A1.1a)}$$

The value of the land at $t \geq T$ if, hypothetically, it were still undeveloped is

$$\Phi(t) = \max_{\hat{K}(t),\hat{T}(t)} \left[\int_{\hat{T}(t)}^\infty r(u)Q\left(\hat{K}(t)\right)e^{-i(u-t)}du - p\hat{K}(t)e^{-i\left(\hat{T}(t)-t\right)} - \tau_S \int_t^\infty \Phi(u)e^{-i(u-t)}du\right], \quad \text{(A1.1b)}$$

where $\hat{T}(t)$ is the profit-maximizing time to develop the land[13] *conditional on its being undeveloped at time t*, and $\hat{K}(t)$ is defined analogously.

Since the developer in fact develops at t=T, post-development raw site value is independent of his actions. Accordingly, he views post-development raw site value tax payments as lump-sum taxes. Pre-development raw site value, meanwhile, is the market value and does depend on the profit-maximizing T and K.

Define Z(T) to be the sum of post-development raw site value tax payments, discounted to T:

$$Z(T) = \tau_S \int_T^\infty \Phi(u)e^{-i(u-T)}du. \quad \text{(A1.2a)}$$

Substituting (A1.2a) into (A1.1a) yields

$$V(t) = \max_{K,T} \left[\int_T^\infty r(u)Q(K)e^{-i(u-t)}du - pKe^{-i(T-t)} - \tau_S \int_t^T V(u)e^{-i(u-t)}du - Z(T)e^{-i(T-t)}\right]. \quad \text{(A1.1a}')$$

Differentiation with respect to t gives

$$\dot{V} = \left(i + \tau_S\right)V$$

and

$$V(t) = \max_{K,T} \left[\int_T^\infty r(u)Q(K)e^{-i(u-T)}du - pK - Z(T)\right]e^{-(i+\tau_S)(T-t)}. \quad (A1.1a'')$$

It is easy to see from (A1.1a″) that the first-order condition for density is independent of τ_S. The first-order condition for development timing is

$$-(i+\tau_S)V(T) - r(T)Q(K) + i\int_T^\infty r(u)Q(K)e^{-i(u-T)}du - Z'(T) = 0. \quad (A1.3a)$$

Substituting (A1.1a″) evaluated at T into (A1.3a) yields

$$-\tau_S V(T) - r(T)Q(K) + ipK + iZ(T) - Z'(T) = 0. \quad (A1.3b),$$

Finally, substituting $Z'(T) = -\tau_S \Phi(T) + iZ(T)$ (from (A1.2a)) and $V(T) = \Phi(T)$, (A1.3b) reduces to the first-order condition without the raw site value tax. Thus, the raw site value tax is neutral.

To understand this result, consider first the development density condition. Turn to (A1.1a′). For the development density condition to be unaffected by the site value tax, the derivative of the last two terms with respect to K must equal zero. The derivative of the second last term equals zero since V(u) is the maximized value of discounted net revenue with respect to K, implying $\frac{\partial V(u)}{\partial K} = 0$. And the derivative of the last term equals zero since, with development time fixed, post-development raw site value tax payments are independent of development density. Consider next the development timing condition. For this condition too to be unaffected by the site value tax, the derivative of the last two terms of (A1.1a′) with respect to T must equal zero. Using (A1.2a), this derivative is

$$-\tau_S \int_t^T \frac{\partial V(u)}{\partial T} e^{-i(u-t)}du - \tau_S (V(T) - \Phi(T))e^{-i(T-t)}.$$

The first term equals zero since $V(u)$ is the maximized value of discounted net revenue with respect to T, implying $\frac{\partial V(u)}{\partial T} = 0$, and the second equals zero since $V(T) = \Phi(T)$.

[13] In a smoothly growing economy, the land would be developed immediately.

Appendix 2

Relationship between τ_V, τ_S, and τ_K
(Not for Publication)

The two neutrality conditions are (21b) and (22). We shall derive the revenue condition. Then we shall have three equations in three unknowns, and we shall investigate the properties of the equation system. We assume throughout that $\tau_S > -i$.

a) the revenue condition

To simplify the notation somewhat, let t=0 denote the time at which the land value tax is first applied.

Let $\mathsf{R}(T)$ denote the value of revenue collected, evaluated at time T:

$$\mathsf{R}(T) = \tau_V \int_0^T V(t)e^{i(T-t)}dt + \tau_K \int_T^\infty pKe^{-i(t-T)}dt + \tau_S \int_T^\infty S(t)e^{-i(t-T)}dt. \qquad \text{(A2.1)}$$

From (15)

$$\begin{aligned}\tau_V \int_0^T V(t)e^{i(T-t)}dt &= \int_0^T \tau_V V(T)e^{-(i+\tau_V)(T-t)}e^{i(T-t)}dt \\ &= V(T)\left(1 - e^{-\tau_V T}\right).\end{aligned} \qquad \text{(A2.2)}$$

From (12)

$$\tau_S \int_T^\infty S(t)e^{-i(t-T)}dt = -S(T) + \int_T^\infty \left(rQ - (i+\tau_K)pK\right)e^{-i(t-T)}dt. \qquad \text{(A2.3)}$$

And

$$\int_T^\infty pKe^{-i(t-T)}dt = \frac{pK}{i}. \qquad \text{(A2.4)}$$

Combining (A2.1) - (A2.4) yields

$$\begin{aligned}\mathsf{R}(T) &= V(T)\left(1 - e^{-\tau_V T}\right) + \frac{\tau_K pK}{i} - S(T) + \int_T^\infty \left(rQ - (i+\tau_K)pK\right)e^{-i(t-T)}dt \\ &= -V(T)e^{-\tau_V T} + \int_T^\infty rQe^{-i(t-T)}dt - pK \qquad \text{(using } S(T) = V(T)) \\ &= -\left(\int_T^\infty \left(rQ - (i+\tau_K)pK\right)e^{-(i+\tau_S)(t-T)}dt\right)e^{-\tau_V T} + \int_T^\infty rQe^{-i(t-T)}dt - pK. \text{ (using (15b))} \qquad \text{(A2.5)}\end{aligned}$$

b) <u>the three equations</u>

Define

$$A \equiv \int_T^\infty r(t)e^{-i(t-T)}dt \qquad B(\tau_S) \equiv \int_T^\infty r(t)e^{-(i+\tau_S)(t-T)}dt. \tag{A2.6}$$

Rewrite the three equations using (A2.6). Eq. (19) becomes

$$A(i+\tau_K)-(i+\tau_S)B(\tau_S)=0. \tag{A2.7}$$

Equation (18b) becomes

$$\tau_K pK + (\tau_S - \tau_V)\left(B(\tau_S)Q - \frac{i+\tau_K}{i+\tau_S}pK\right)=0. \tag{A2.8}$$

And eq. (A2.5) becomes

$$\mathsf{R}(T) = -\left(B(\tau_S)Q - \left(\frac{i+\tau_K}{i+\tau_S}\right)pK\right)e^{-\tau_V T} + AQ - pK = 0. \tag{A2.9}$$

Now substitute (A2.7) into (A2.8) and (A2.9). Then the three equations can be written as

$$A(i+\tau_K)-(i+\tau_S)B(\tau_S)=0 \tag{A2.7}$$

$$\tau_K pK + (\tau_S - \tau_V)\left(\frac{i+\tau_K}{i+\tau_S}\right)(AQ-pK)=0 \tag{A2.8´}$$

$$\mathsf{R}(T) = (AQ-pK)\left(1-\frac{i+\tau_K}{i+\tau_S}e^{-\tau_V T}\right). \tag{A2.9´}$$

Note that AQ-pK is site value at development time in the pre-tax situation. Thus,

$$\varepsilon \equiv \frac{\mathsf{R}(T)}{AQ-pK} \tag{A2.10}$$

is the ratio of the value of tax collected evaluated at development time to the pre-tax site value at development time, which is the measure employed of the proportion of site value expropriated through the tax. Define $\tilde{\eta}$ implicitly as

$$i-\tilde{\eta} = r(T)/A. \tag{A2.11}$$

Then the development timing condition in the pre-tax situation (eq. 2) can be written as

$A(i-\tilde{\eta})Q - ipK = 0$, so that

$$AQ - pK = pK\left(\frac{i}{i-\tilde{\eta}} - 1\right) = \frac{pK\tilde{\eta}}{i-\tilde{\eta}}. \tag{A2.12}$$

Using (A2.10) and (A2.12), (A2.7), (A2.8´), and (A2.9´) can be rewritten as

$$A(i+\tau_K) - (i+\tau_S)B(\tau_S) = 0 \tag{A2.7}$$

$$\tau_K + (\tau_S - \tau_V)\left(\frac{i+\tau_K}{i+\tau_S}\right)\frac{\tilde{\eta}}{i-\tilde{\eta}} = 0 \tag{A2.8″}$$

$$1 - \left(\frac{i+\tau_K}{i+\tau_S}\right)e^{-\tau_V T} - \varepsilon = 0. \tag{A2.9″}$$

This set of three equations characterizes the set of (τ_V, τ_S, and τ_K) that achieve neutrality and expropriate a proportion ε of site value.

c) $\underline{\tau_K = \tau_K(\tau_S)}$

Observe that (A2.7) gives τ_K as a function of τ_S. When τ_S=0, A=B(τ_S) so that τ_K=0; thus, τ_K(τ_S) passes through the origin in τ_S-τ_K space. Also,

$$\left.\frac{d\tau_K}{d\tau_S}\right|_{(A2.7)} = \frac{B(\tau_S) + (i+\tau_S)B'(\tau_S)}{A}. \tag{A2.13}$$

From (A2.6), defining u=t-T, $B(\tau_S) = \int_0^\infty r(u)e^{-(i+\tau_S)u}du$, so that

$$B'(\tau_S) = -\int_0^\infty r(u)ue^{-(i+\tau_S)u}du$$

$$= \left(\frac{r(u)ue^{-(i+\tau_S)u}}{i+\tau_S}\right)_0^\infty - \int_0^\infty \frac{(\dot{r}u + r)}{(i+\tau_S)}e^{-(i+\tau_S)u}du < 0 \text{ (integration by parts).} \tag{A2.14}$$

Substituting (A2.14) into (A2.13) yields

$$\left.\frac{d\tau_K}{d\tau_S}\right|_{(A2.7)} = -\frac{1}{A}\int_0^\infty \dot{r}ue^{-(i+\tau_S)u}du. \tag{A2.15}$$

This is ambiguous in sign. In a growing economy, however, one expects $\dot{r} > 0$, except for downturns in the business cycle. Thus, "normally" $\left.\frac{d\tau_K}{d\tau_S}\right|_{(A2.7)} < 0$.

(AS-1): $\int_0^\infty \dot{r} u e^{-(i+\tau_S)u} du > 0$ for $u \equiv t - T$ and $\tau_S > -i$

Proposition A1: Under (AS-1), $\left.\frac{d\tau_K}{d\tau_S}\right|_{(A2.7)} < 0$.

Differentiating (A2.15) with respect to τ_S gives

$$\left.\frac{d^2\tau_K}{d\tau_S{}^2}\right|_{(A2.7)} = \frac{1}{A}\int_0^\infty \dot{r} u^2 e^{-(i+\tau_S)u} du.$$

(AS-2): $\int_0^\infty \dot{r} u^2 e^{-(i+\tau_S)u} du > 0$ for $u \equiv t - T$ and $\tau_S > -i$

Proposition A2: Under (AS-2), $\left.\frac{d^2\tau_K}{d^2\tau_S}\right|_{(A2.7)} > 0$.

Figure A1 plots the relationship between τ_K and τ_S under (AS-1) and (AS-2).

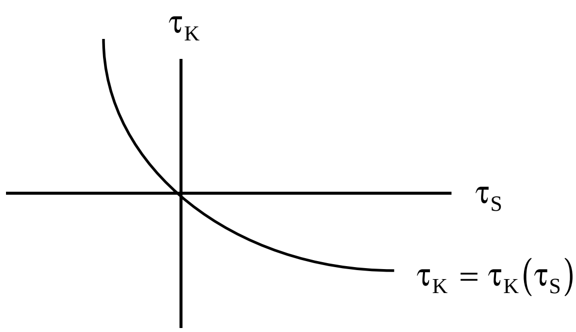


Figure A1

d) $\tau_V = \bar{\tau}_V(\tau_S)$

Substituting (A2.7) into (A2.9″) gives an implicit relationship between τ_V and $\tau_S\left(\tau_V = \bar{\tau}_V(\tau_S)\right)$:

$$1 - \frac{B(\tau_S)}{A} e^{-\tau_V T} - \varepsilon = 0. \tag{A2.16}$$

Differentiating (A2.16) yields

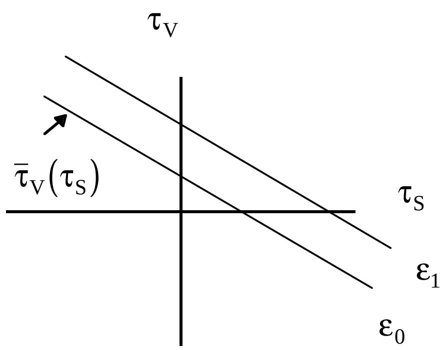
$$\left.\frac{d\tau_V}{d\tau_S}\right|_{(A2.16)} = \frac{B'}{BT} < 0 \tag{A2.17}$$

Also,

$$\left.\frac{d^2\tau_V}{d\tau_S{}^2}\right|_{(A2.16)} = \frac{B''}{BT} - \frac{(B')^2}{B^2T},$$

which is in general ambiguous in sign.

Figure A2 plots this relationship between τ_V and τ_S.



$\varepsilon_1 > \varepsilon_0$

Figure A2

Loci further northeast from the origin correspond to higher levels of expropriation.

e) $\underline{\tau_V = \bar{\bar{\tau}}_V(\tau_S)}$

Substituting (A2.7) into (A2.8″) gives another relationship between τ_V and $\tau_S\left(\tau_V = \bar{\bar{\tau}}_V(\tau_S)\right)$:

$$\left(\frac{(i+\tau_S)B(\tau_S)}{A} - i\right) + (\tau_S - \tau_V)\frac{B(\tau_S)}{A}\frac{\tilde{\eta}}{i-\tilde{\eta}} = 0$$

or

$$\tau_V = \frac{i(i-\tilde{\eta})}{\tilde{\eta}}\left(1 - \frac{A}{B(\tau_s)}\right) + \tau_S\frac{i}{\tilde{\eta}}. \qquad \text{(A2.18)}$$

Note first that when τ_S=0, A=B, which implies that τ_V=0. Thus, $\bar{\bar{\tau}}_V(\tau_S)$ passes through the origin in $\tau_S - \tau_K$ space. Also,

$$\left.\frac{d\tau_V}{d\tau_S}\right|_{(A2.18)} = \frac{i}{\tilde{\eta}}\left((i-\tilde{\eta})\frac{A}{B^2}B' + 1\right). \qquad \text{(A2.19)}$$

Now define $\hat{\eta}(\tau_S)$ implicitly by

$$B = \int_0^\infty r(u)e^{-(i+\tau_S)u}du = \int_0^\infty r(0)e^{-(i+\tau_S-\hat{\eta}(\tau_S))u}du \qquad \text{(A2.20a)}$$

and $\hat{\hat{\eta}}(\tau_S)$ implicitly by

$$B' = -\int_0^\infty r(u)ue^{-(i+\tau_S)u}du = -\int_0^\infty r(0)ue^{-\left(i+\tau_S-\hat{\hat{\eta}}(\tau_S)\right)u}du. \qquad \text{(A2.20b)}$$

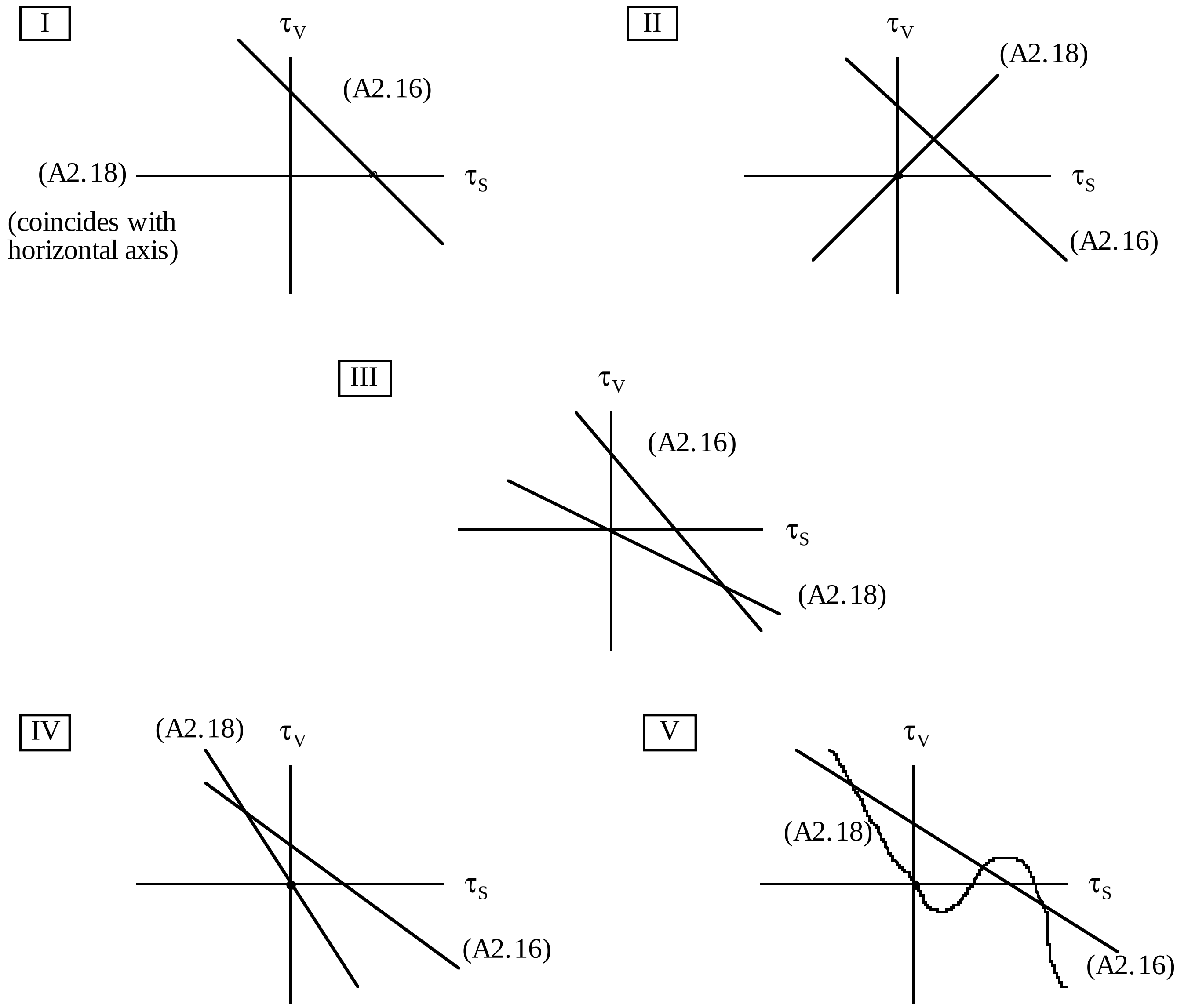
Both $\hat{\eta}(\tau_S)$ and $\hat{\hat{\eta}}(\tau_S)$ are weighted average rental growth rates. Because B' contains the extra u inside the integral, the calculation of $\hat{\hat{\eta}}(\tau_S)$ puts more weight on later periods than does $\hat{\eta}(\tau_S)$. Thus $\hat{\eta}(\tau_S) \gtrless \hat{\hat{\eta}}(\tau_S)$ if the rental growth rate $\begin{pmatrix}\text{falls}\\ \text{rises}\end{pmatrix}$ over time. Substituting (A2.20a), (A2.20b), and the definitions of $\hat{\eta}$ and $\hat{\hat{\eta}}$ into (A2.19) yields

$$\left.\frac{d\tau_V}{d\tau_S}\right|_{(A2.18)} = \frac{i}{\tilde{\eta}}\left(-\left(\frac{i+\tau_S-\hat{\eta}(\tau_S)}{i+\tau_S-\hat{\hat{\eta}}(\tau_S)}\right)^2+1\right). \qquad \text{(A2.21)}$$

If rental growth is exponential, $\hat{\eta}(\tau_S)=\hat{\hat{\eta}}(\tau_S)$ so that $\left.\frac{d\tau_V}{d\tau_S}\right|_{(A2.18)} = 0$. If rental growth $\begin{pmatrix}\text{falls}\\ \text{rises}\end{pmatrix}$ over time, $\left.\frac{d\tau_V}{d\tau_S}\right|_{(A2.18)} \gtrless 0$, and when rental growth fluctuates over time $\bar{\bar{\tau}}_V(\tau_S)$ need not be monotonic. These results are sufficiently important that we record them.

<u>Proposition A3</u>: $\left.\frac{d\tau_V}{d\tau_S}\right|_{(A2.18)}$ has the same sign as $\hat{\eta}(\tau_S)-\hat{\hat{\eta}}(\tau_S)$. Thus, if $\hat{\eta}(\tau_S) \begin{matrix}>\\=\\<\end{matrix} \hat{\hat{\eta}}(\tau_S)$ for all τ_S, then $\bar{\bar{\tau}}_V(\tau_S)$ is $\begin{Bmatrix}\text{monotonically increasing}\\ \text{constant}\\ \text{monotonically decreasing}\end{Bmatrix}$.

Figure A3 plots $\bar{\tau}_V(\tau_S)$ ((A2.16)) and $\bar{\bar{\tau}}_V(\tau_S)$ ((A2.18)) for five cases.



Case I depicts the situation with exponential rental growth. Neutrality entails $\tau_S > 0$, $\tau_V = 0$, and (from Figure A.1 since (AS-1) is satisfied) $\tau_K < 0$. Case II depicts a maturing city in which the growth rate of rents is positive but falling over time. Neutrality entails $\tau_S > 0$, $\tau_V > 0$, and $\tau_K < 0$. Cases III and IV depict incipient boom towns in which the rental growth rate is increasing over time. In Case III, $\tau_S > 0$, $\tau_V < 0$, and (when (AS-1) is satisfied) $\tau_K < 0$; and in case IV, $\tau_S < 0$, $\tau_V > 0$, and (when (AS-1) is satisfied)

$\tau_K > 0$. Case V demonstrates the possibility of multiple neutral property tax systems satisfying a particular revenue requirement.

The above line of analysis can be extended straightforwardly to more complex situations, for example where the interest rate varies over time and where there is technical change in construction. The extension to treat uncertainty — for example where the time path of rents follows a stochastic process — will be more difficult.

Appendix 1

Proof of Result 3:

Prior to development, raw site value equals the value of the vacant land, $V(t)$. After development, raw site value equals the value of the land were it still undeveloped.

Using (6), the value of vacant land for t<T is

$$V(t) = \max_{K,T} \left[\int_T^\infty r(u)Q(K)e^{-i(u-t)}du - pKe^{-i(T-t)} - \tau_S \int_t^\infty S(u)e^{-i(u-t)}du\right]$$

$$= \max_{K,T} \left[\int_T^\infty r(u)Q(K)e^{-i(u-t)}du - pKe^{-i(\tau-t)} - \tau_S \int_t^T V(u)e^{-i(u-t)}du - \tau_S \int_T^\infty \Phi(u)e^{-i(u-t)}du\right]. \quad (8a)$$

The value of the land at $t \geq T$ if, hypothetically, it were still undeveloped is

$$\Phi(t) = \max_{\hat{K}(t),\hat{T}(t)} \left[\int_{\hat{T}(t)}^\infty r(u)Q(K(t))e^{-i(u-t)}du - p\hat{K}(t)e^{-i(\hat{T}(t)-t)} - \tau_S \int_t^\infty \Phi(u)e^{-i(u-t)}du\right], \quad (8b)$$

where $\hat{T}(t)$ is the profit-maximizing time to develop the land[1] *conditional on its being undeveloped at time t*, and $\hat{K}(t)$ is defined analogously.

Since the developer in fact develops at t=T, post-development raw site value is independent of his actions. Accordingly, he views post-development raw site value tax payments as lump-sum taxes. Pre-development raw site value, meanwhile, is the market value and does depend on the profit-maximizing T and K.

Define Z(T) to be the sum of post-development raw site value tax payments, discounted to T:

$$Z(T) = \tau_S \int_T^\infty \Phi(u)e^{-i(u-T)}du. \quad (9a)$$

Substituting (9a) into (8a) yields

$$V(t) = \max_{K,T} \left[\int_T^\infty r(u)Q(K)e^{-i(u-t)}du - pKe^{-i(T-t)} - \tau_S \int_t^T V(u)e^{-i(u-t)}du - Z(T)e^{-i(T-t)}\right]. \quad (8a')$$

Differentiation with respect to t gives

$$\dot{V} = \left(i + \tau_S\right)V$$

and

$$V(t) = \max_{K,T} \left[\int_T^{\infty} r(u)Q(K)e^{-i(u-T)}du - pK - Z(T)\right]e^{-\left(i+\tau_S\right)(T-t)} \qquad (8a'')$$

It is easy to see from $(8a'')$ that the first-order condition for density is independent of τ_S. The first-order condition for development timing is

$$-\left(i + \tau_S\right)V(t) + \left[-r(T)Q(K) + i\int_T^{\infty} r(u)Q(K)e^{-i(u-T)}du - Z'(T)\right]e^{-\left(i-\tau_S\right)(T-t)} = 0. \qquad (10a)$$

Substituting $(8a'')$ into (10a) and evaluating at t=T yields

$$-\tau_S V(T) - r(T)Q(K) + ipK + iZ(T) - Z'(T) = 0. \qquad (10b),$$

Finally, substituting $Z'(T) = -\tau_S \Phi(T) + iZ(T)$ (from (9a)) and $V(T) = \Phi(T)$, (10b) reduces to the first-order condition without the raw site value tax. Thus, the raw site value tax is neutral. [1]

To understand this result, consider first the development density condition. Turn to $(8a')$. For the development density condition to be unaffected by the site value tax, the derivative of the last two terms with respect to K must equal zero. The derivative of the second last term equals zero since V(u) is the maximized value of discounted net revenue with respect to K, implying $\frac{\partial V(u)}{\partial K} = 0$. And the derivative of the last term equals zero since, with development time fixed, post development raw site value tax payments are independent of development density. Consider next the development timing condition. For this condition too to be unaffected by the site value tax, the derivative of the last two terms of $(8a')$ with respect to T must equal zero. Using (9a), this derivative is

$$-\tau_S \int_t^T \frac{\partial V(u)}{\partial T} e^{-i(u-t)}du - \tau_S \left(V(T) - \Phi(T)\right)e^{-i(T-t)}.$$

The first term equals zero since $V(u)$ is the maximized value of discounted net revenue with

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[1] In a smoothly growing economy, the land would be developed immediately.

respect to T, implying $\frac{\partial V(u)}{\partial T} = 0$, and the second equals zero since $V(T) = \Phi(T)$.

Appendix 2

Relationship between τ_V, τ_S, and τ_K
(Not for Publication)

The two neutrality conditions are (21b) and (22). We shall derive the revenue condition. Then we shall have three equations in three unknowns, and we shall investigate the properties of the equation system. We assume throughout that $\tau_S < i$

a) the revenue condition

To simplify the notation somewhat, let t=0 denot the time at which the land value systems is applied.

Let R (T) denote the value of revenue collected, evaluated at time T:

$$\mathsf{R}(T) = \tau_V \int_0^T V(t)e^{i(T-t)}dt + \tau_K \int_T^\infty pKe^{-i(t-T)}dt + \tau_S \int_T^\infty S(t)e^{-i(t-T)}dt. \tag{A.1}$$

From (18)

$$\begin{aligned}\tau_V \int_0^T V(t)e^{i(T-t)}dt &= \int_0^T \tau_V V(T)e^{-(i+\tau_V)(T-t)}e^{i(T-t)}dt \\ &= V(T)\left(1-e^{-\tau_V T}\right).\end{aligned} \tag{A.2}$$

From (15)

$$\tau_S \int_T^\infty S(t)e^{-i(t-T)}dt = -S(T) + \int_T^\infty \left(rQ - \left(i+\tau_K\right)pK\right)e^{-i(t-T)}dt. \tag{A.3}$$

And

$$\int_T^\infty pKe^{-i(t-T)}dt = \frac{pK}{i}. \tag{A.4}$$

Combining (A.1) - (A.4) yields

$$\begin{aligned}\mathsf{R}(T) &= V(T)\left(1-e^{-\tau_V T}\right) + \frac{\tau_K pK}{i} - S(T) + \int_T^\infty \left(rQ - \left(i+\tau_K\right)pK\right)e^{-i(t-T)}dt \\ &= -V(T)e^{-\tau_V T} + \int_T^\infty rQe^{-i(t-T)}dt - pK \qquad \text{(using } S(T) = V(T)\text{)} \\ &= -\left(\int_T^\infty \left(rQ - \left(i+\tau_K\right)pK\right)e^{-(i+\tau_S)(t-T)}dt\right)e^{-\tau_V T} + \int_T^\infty rQe^{-i(t-T)}dt - pK. \quad \text{(using (18b)}\end{aligned} \tag{A.5}$$

b) <u>the three equations</u>

Define

$$A \equiv \int_T^\infty r(t)e^{-i(t-T)}dt \qquad\qquad B(\tau_S) \equiv \int_T^\infty r(t)e^{-(i+\tau_S)(t-T)}dt. \tag{A.6}$$

Rewrite the three equations using (A.6). Eq. (22) becomes

$$A(i+\tau_K)-(i+\tau_S)B(\tau_S)=0. \tag{A.7}$$

Equation (21b) becomes

$$\tau_K pK + (\tau_S-\tau_V)\left(B(\tau_S)Q - \frac{i+\tau_K}{i+\tau_S}pK\right)=0. \tag{A.8}$$

And eq. (A.5) becomes

$$\mathsf{R}(T) = -\left(B(\tau_S)Q-\left(\frac{i+\tau_K}{i+\tau_S}\right)pK\right)e^{-\tau_V T}+AQ-pK=0. \tag{A.9}$$

Now substitute (A.7) into (A.8) and (A.9). Then the three equations can be written as

$$A(i+\tau_K)-(i+\tau_S)B(\tau_S)=0 \tag{A.7}$$

$$\tau_K pK + (\tau_S-\tau_V)\left(\frac{i+\tau_K}{i+\tau_S}\right)(AQ-pK)=0 \tag{A.8´}$$

$$\mathsf{R}(T) = (AQ-pK)\left(1-\frac{i+\tau_K}{i+\tau_S}e^{-\tau_V T}\right). \tag{A.9´}$$

Note that AQ-pK is site value at development time in the pre-tax situation. Thus,

$$\varepsilon \equiv \frac{\mathsf{R}(T)}{AQ-pK} \tag{A.10}$$

is the ratio of the value of tax collected evaluated at development time to the pre-tax site value at development time, which is the measure employed of the proportion of site value expropriated through the tax. Define η implicitly as

$$i-\eta = r(T)/A. \tag{A.11}$$

Then the development timing condition in the pre-tax situation (eq. 2) can be written as

$A(i-\eta)Q - ipK = 0$, so that

$$AQ - pK = pK\left(\frac{i}{i-\eta} - 1\right) = \frac{pK\eta}{i-\eta}. \tag{A.12}$$

Using (A.10) and (A.12), (A.7), (A.8´), and (A.9´) can be rewritten as

$$A(i+\tau_K) - (i+\tau_S)B(\tau_S) = 0 \tag{A.7}$$

$$\tau_K + (\tau_S - \tau_V)\left(\frac{i+\tau_K}{i+\tau_S}\right)\frac{\eta}{i-\eta} = 0 \tag{A.8″}$$

$$1 - \left(\frac{i+\tau_K}{i+\tau_S}\right)e^{-\tau_V T} - \varepsilon = 0. \tag{A.9″}$$

This set of three equations characterizes the set of (τ_V, τ_S, and τ_K) that achieve neutrality and expropriate a proportion ε of site value.

c) $\underline{\tau_K = \tau_K(\tau_S)}$

Observe that (A.7) gives τ_K as a function of τ_S. When τ_S=0, A=B(τ_S) so that τ_K=0; thus, τ_K(τ_S) passes through the origin in τ_S-τ_K space. Also,

$$\left.\frac{d\tau_K}{d\tau_S}\right|_{(A.7)} = \frac{B(\tau_S) + (i+\tau_S)B'(\tau_S)}{A}. \tag{A.13}$$

From (A.6), defining u=t-T, $B(\tau_S) = \int_0^\infty r(u)e^{-(i+\tau_S)u}du$, so that

$$B'(\tau_S) = -\int_0^\infty r(u)ue^{-(i+\tau_S)u}du$$

$$= \left(\frac{r(u)ue^{-(i+\tau_S)u}}{i+\tau_S}\right)_0^\infty - \int_0^\infty \frac{(\dot{r}u + r)}{(i+\tau_S)}e^{-(i+\tau_S)u}du < 0. \text{ (integration by parts)} \tag{A.14}$$

Substituting (A.14) into (A.13) yields

$$\left.\frac{d\tau_K}{d\tau_S}\right|_{(A.7)} = -\frac{1}{A}\int_0^\infty \dot{r}ue^{-(i+\tau_S)u}du. \tag{A.15}$$

This is ambiguous in sign. In a growing economy, however, one expects $\dot{r} > 0$, except for downturns in the business cycle. Thus, "normally" $\left.\frac{d\tau_K}{d\tau_S}\right|_{(A.7)} < 0$.

(AS-1): $\int_0^\infty \dot{r} u e^{-(i+\tau_S)u} du > 0$ for $u \equiv t - T$ and $\tau_S > -i$

Proposition A1: Under (AS-1), $\left.\frac{d\tau_K}{d\tau_S}\right|_{(A.7)} < 0$.

Differentiating (A.15) with respect to τ_S gives

$$\left.\frac{d^2\tau_K}{d\tau_S^2}\right|_{(A.7)} = \frac{1}{A}\int_0^\infty \dot{r} u^2 e^{-(i+\tau_S)u} du.$$

(AS-2): $\int_0^\infty \dot{r} u^2 e^{-(i+\tau_S)} du > 0$ for $u \equiv t - T$ and $\tau_S > -i$

Proposition A2: Under (AS-2), $\left.\frac{d^2\tau_K}{d^2\tau_S}\right|_{(A.7)} < 0$.

Figure A1 plots the relationship between τ_K and τ_S under (AS-1) and (AS-2).

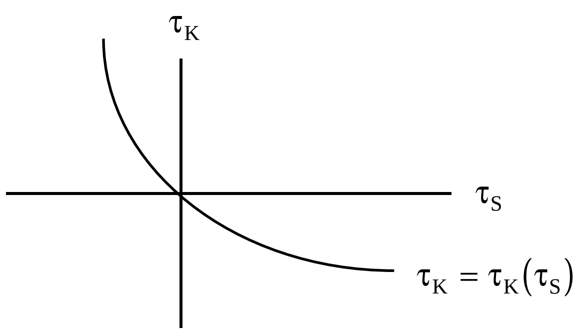


Figure A1

d) $\underline{\tau_V = \bar{\tau}_V(\tau_S)}$

Substituting (A.7) into (A.9″) gives an implicit relationship between τ_V and $\tau_S\left(\tau_V = \bar{\tau}_V(\tau_S)\right)$:

$$1 - \frac{B(\tau_S)}{A} e^{-\tau_V T} - \varepsilon = 0. \tag{A.16}$$

Differentiating (A.16) yields

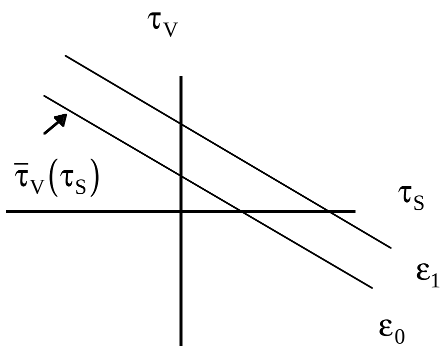
$$\left.\frac{d\tau_V}{d\tau_S}\right|_{(A.16)} = \frac{B'}{BT} < 0 \tag{A.17}$$

Also,

$$\left.\frac{d^2\tau_V}{d\tau_S^2}\right|_{(A.16)} = \frac{B''}{BT} - \frac{(B')^2}{B^2 T},$$

which is in general ambiguous in sign.

Figure A2 plots this relationship between τ_V and τ_S.



$\varepsilon_1 > \varepsilon_0$

Figure A2

Loci further northeast from the origin correspond to higher levels of expropriation.

e) $\underline{\tau_V = \bar{\bar{\tau}}_V(\tau_S)}$

Substituting (A.7) into (A. 8″) gives another relationship between τ_V and $\tau_S\left(\tau_V = \bar{\bar{\tau}}_V(\tau_S)\right)$:

$$\left(\frac{(i+\tau_S)B(\tau_S)}{A} - i\right) + (\tau_S - \tau_V)\frac{B(\tau_S)}{A}\frac{\eta}{i-\eta} = 0$$

or

$$\tau_V = \frac{i(i-\eta)}{\eta}\left(1 - \frac{A}{B(\tau_s)}\right) + \tau_S\frac{i}{\eta}. \qquad \text{(A.18)}$$

Note first that when τ_S=0, A=B, which implies that τ_V=0. Thus $\bar{\bar{\tau}}_V(\tau_S)$ passes through the origin in $\tau_S - \tau_K$ space. Also,

$$\left.\frac{d\tau_V}{d\tau_S}\right|_{(A.18)} = \frac{i}{\eta}\left((i-\eta)\frac{A}{B^2}B' + 1\right). \qquad \text{(A.19)}$$

Now define $\hat{\eta}(\tau_S)$ implicitly by

$$B = \int_0^\infty r(u)e^{-(i+\tau_S)u}du = \int_0^\infty r(0)e^{-(i+\tau_S-\hat{\eta}(\tau_S))u}du \qquad \text{(A.20a)}$$

and $\hat{\hat{\eta}}(\tau_S)$ implicitly by

$$B' = -\int_0^\infty r(u)ue^{-(i+\tau_S)u}du = -\int_0^\infty r(0)ue^{-\left(i+\tau_S-\hat{\hat{\eta}}(\tau_S)\right)u}du. \tag{A.20b}$$

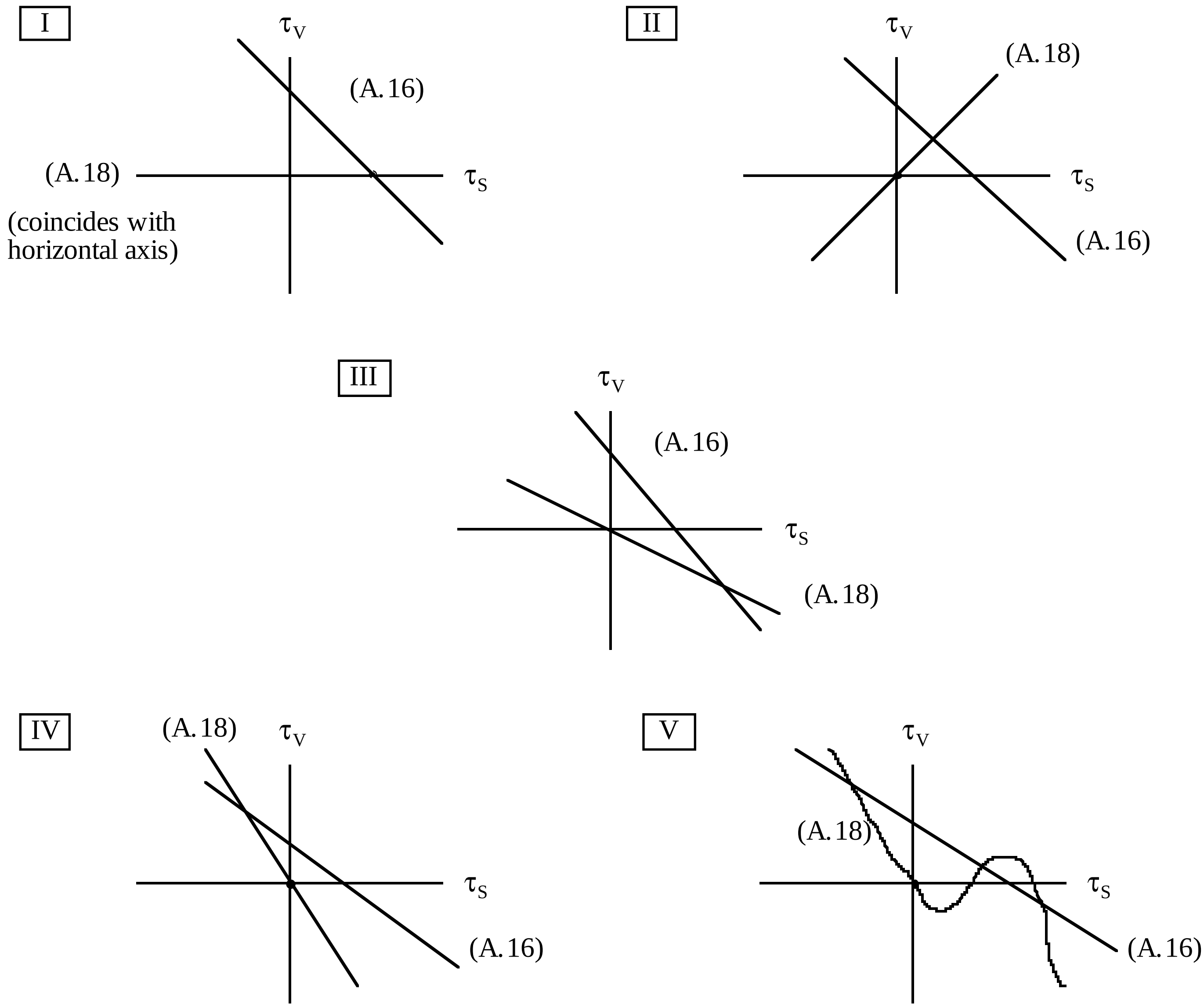
Both $\hat{\eta}(\tau_S)$ and $\hat{\hat{\eta}}(\tau_S)$ are average rental growth rates. Because B' contains the extra u inside the integral, the calculation of $\hat{\hat{\eta}}(\tau_S)$ puts more weight on later periods than does $\hat{\eta}(\tau_S)$. Thus $\hat{\eta}(\tau_S) \gtrless \hat{\hat{\eta}}(\tau_S)$ if the rental growth rate $\begin{pmatrix}\text{falls}\\ \text{rises}\end{pmatrix}$ over time. Substituting (A.20a), (A.20b), and the definition of η into (A.19) yields

$$\left.\frac{d\tau_V}{d\tau_S}\right|_{(A.18)} = \frac{i}{\eta}\left(-\left(\frac{i+\tau_S-\hat{\eta}(\tau_S)}{i+\tau_S-\hat{\hat{\eta}}(\tau_S)}\right)^2 + 1\right). \tag{A.21}$$

If rental growth is exponential, $\hat{\eta}(\tau_S) = \hat{\hat{\eta}}(\tau_S)$ so that $\left.\frac{d\tau_V}{d\tau_S}\right|_{(A.18)} = 0$. If rental growth $\begin{pmatrix}\text{falls}\\ \text{rises}\end{pmatrix}$ over time, $\left.\frac{d\tau_V}{d\tau_S}\right|_{(A.18)} \gtrless 0$, and when rental growth fluctuates over time $\bar{\bar{\tau}}_V(\tau_S)$ need not be monotonic. These results are sufficiently important that we record them.

<u>Proposition A3</u>: $\left.\frac{d\tau_V}{d\tau_S}\right|_{(A.18)}$ has the same sign as $\hat{\eta}(\tau_S) - \hat{\hat{\eta}}(\tau_S)$. Thus, if $\hat{\eta}(\tau_S) \begin{smallmatrix}>\\=\\<\end{smallmatrix} \hat{\hat{\eta}}(\tau_S)$ for all τ_S, then $\bar{\bar{\tau}}_V(\tau_S)$ is $\begin{Bmatrix}\text{monotonically increasing}\\ \text{constant}\\ \text{monotonically decreasing}\end{Bmatrix}$.

Figure A3 plots $\bar{\tau}_V(\tau_S)$ ((A.16)) and $\bar{\bar{\tau}}_V(\tau_S)$ ((A.18)) for five cases.



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$\tau_K > 0$. Case V demonstrates the possibility of multiple neutral property tax systems satisfying a particular revenue requirement.

The above line of analysis can be extended straightforwardly to more complex situations, for example where the interest rate varies over time and where there is technical change in construction. The extension to treat uncertainty — for example where the time path of rents follows a stochastic process — will be more difficult.