Economics Department

Working Papers in Economics

Boston College *Year* 2001

Urban Economic Aggregates in Monocentric and Non-monocentric Cities

Richard Arnott
Boston College,

Urban Economic Aggregates
in
Monocentric and Non-monocentric Cities

Richard Arnott*

May 2001

* Department of Economics
Boston College
Chestnut Hill. MA 02467
USA
617-552-3674
richard.arnott@bc.edu

Urban Economic Aggregates in Monocentric and Non-monocentric Cities

1. Introduction

It is well known that there is an elegant pair of relationships between urban economic aggregates in simple, *monocentric* cities. The first concerns the relationship between aggregate transport costs (ATC) and differential land rents (DLR – the excess of urban land rents over the agricultural land rent), the second the relationship between urban economic aggregates in cities of optimal population size.

The monocentric city model has been under attack in recent years as providing a twentieth-century analysis of the nineteenth-century city, since over the course of the twentieth century cities have become increasingly polycentric. This paper addresses the question: *How do the relationships between urban economic aggregates in monocentric cities extend to polycentric cities?*

This is an apt choice of topic for this workshop since — as will become evident — it is clearly related to Rena Sivitaidou's first published paper (with Bill Wheaton), "Wage and Rent Capitalization in the Commercial Real Estate Market" (*Journal of Urban Economics* 31, 206-229 (1992)), though its focus is different.

Section 2 reviews the relationship between ATC and DLR in monocentric cities. Section 3 explores how this relationship extends to non-monocentric cities. Section 4 examines the relationship between urban economic aggregates in cities of optimal population size in both monocentric and non-monocentric cities. And section 5 concludes.

2. ATC and DLR in Monocentric Cities

The first paper on this topic was Mohring (1961). Mohring examined a circular city with linear transport costs in which all individuals have identical and fixed lot sizes, $\overline{T}$. Let x denote distance from the central business district (CBD), t transport costs per unit distance, fixed lot size, *R(x)* the rent function, *N* the city's population, and R_A the agricultural rent.

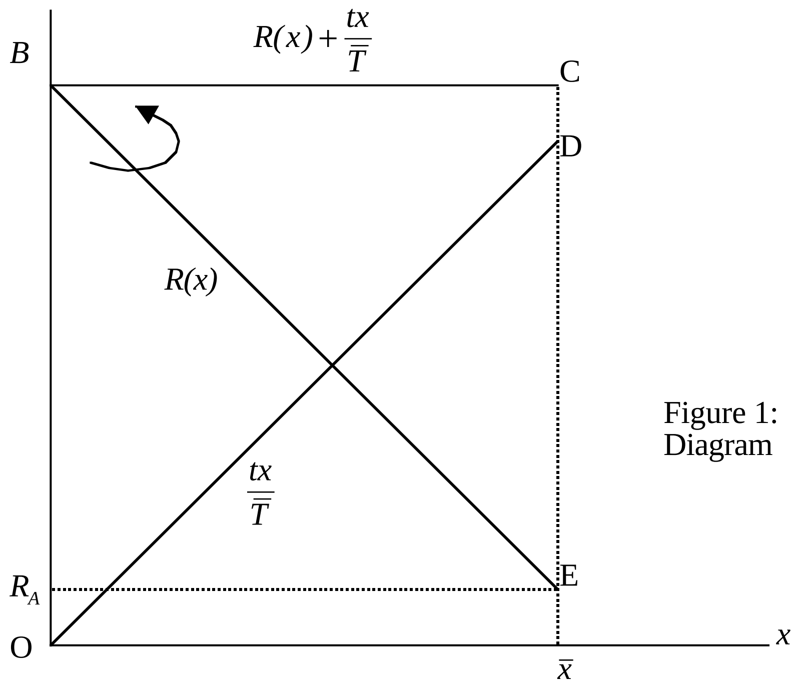

Figure 1: The Mohring Diagram

Consider an individual's choice of residential location. If she moves a unit distance further from the CBD her transport costs increase by *t*. If lot rent falls by less than *t*, she prefers the more central location; if it falls by more than *t*, she prefers the more distant location; and if it falls by exactly *t*, she is indifferent between the two locations. To satisfy the equilibrium condition that the supply of land equal the demand for land at all locations therefore requires that $R'(x)\overline{T} = -t$. It also requires that all households obtain a lot, and that at the urban boundary, $\overline{x}$, the urban rent equals the agricultural rent, $N\overline{T} = \Pi x^2$ and $R(\overline{x}) = R_A$, respectively. Thus, at all locations between the boundary and the CBD

$$R(O)\overline{T} = R(x)\overline{T} + tx = R(\overline{x})\overline{T} + t\overline{x} = R_A\overline{T} + t\overline{x}. \quad (1)$$

This series of equalities is depicted in Figure 1.

Now imagine rotating the diagram around the y-axis. The base is then a circle with radius $\overline{x}$, which is just the area of the city, and the height of BE at a particular location the land rent there. Thus, aggregate land rents in the city are given by the volume of the cone with base radius $\overline{x}$ and height *R(O)*. Thus,

$$\text{ALR} = \frac{1}{3}bh = \frac{1}{3}\left(\pi\bar{x}^2\right)R(O). \tag{2}$$

Differential land rents are given by the corresponding volume but with agricultural land rents subtracted off:

$$\begin{aligned}\text{DLR} &= \frac{1}{3}\left(\pi\bar{x}^2\right)\left(R(O) - R_A\right) \\ &= \frac{1}{3}\left(\pi\bar{x}^2\right)\left(\frac{t\bar{x}}{\overline{\overline{T}}}\right). \qquad \text{(using (1))}\end{aligned} \tag{3}$$

The sum of land rent and transport cost per unit area at location *x* is $R(x) + \frac{tx}{\overline{\overline{T}}} = R_A + \frac{t\bar{x}}{\overline{\overline{T}}}$.

Thus, aggregate land rents in the city plus aggregate transport costs are given by the volume of a cylinder with base radius $\bar{x}$ and height $R_A + \frac{t\bar{x}}{\overline{\overline{T}}}$. Thus

$$\text{ATC} + \text{ALR} = \left(\pi\bar{x}^2\right)\left(R_A + \frac{t\bar{x}}{\overline{\overline{T}}}\right). \tag{4}$$

Subtracting off agricultural rents yields

$$\text{ATC} + \text{DLR} = \left(\pi\bar{x}^2\right)\left(\frac{t\bar{x}}{\overline{\overline{T}}}\right). \tag{5}$$

Comparing (3) and (5) yields

$$\text{ATC=2DLR.} \tag{6}$$

Arnott and Stiglitz (1981— AS hereafter) generalize the result in a number of directions. They demonstrated first that the result generalizes to variable lot size. With variable lot size, the equilibrium condition that land rent be such that individuals are indifferent as to where in the city they locate is the "Muth condition":

$$R'(x) = -\frac{t}{T(x)}, \tag{7}$$

where *T(x)* is the equilibrium lot size at *x*. Then

$$\text{DLR} = \int_0^{\bar{x}} (R(x) - R_A) 2\pi x dx$$

$$= \int_0^{\bar{x}} R(x) 2\pi x dx - \int_0^{\bar{x}} R_A 2\pi x dx$$

$$= \left(R(x)\pi x^2\right)_0^{\bar{x}} - \int_0^{\bar{x}} R'(x)\pi x^2 dx - R_A \pi x^2 \quad \text{(integration by parts)}$$

$$= R(\bar{x})\pi\bar{x}^2 + \int_0^{\bar{x}} \frac{t}{T(x)}\pi x^2 dx - R_A \pi \bar{x}^2 \qquad \text{(using (7))}$$

$$= \frac{1}{2}\int_0^{\bar{x}} \frac{tx}{T(x)} 2\pi x dx \qquad \text{(using } R(\bar{x}) = R_A\text{)}$$

$$= \frac{1}{2}\text{ATC}. \tag{8}$$

The essential step in the derivation is the integration by parts, which shall recur repeatedly in the analysis that follows.

AS then generalized the result to an arbitrary *transport cost function, f(x)*, which gives the cost of transportation associated with location *x*, and an arbitrary *shape of the city, Φ(x)*, which gives the residential land area within a distance *x* of the CBD. With the transport cost function, *f(x)* the Muth rule is

$$R'(x) = -\frac{f'(x)}{T(x)}. \tag{7′}$$

Thus,

$$\text{DLR} = \int_0^{\bar{x}} \left(R(x) - R_A\right)\varphi(x) dx \qquad \left(\varphi(x) \equiv \Phi'(x)\right)$$

$$= -\int_0^{\bar{x}} R'(x)\Phi(x) dx \qquad \text{(integration by parts)}$$

$$= \int_0^{\bar{x}} \frac{f'(x)}{T(x)}\Phi(x) dx. \qquad \text{(using (7′))} \tag{9}$$

Meanwhile

$$\text{ATC} = \int_0^{\bar{x}} \frac{f(x)}{T(x)}\varphi(x) dx. \tag{10}$$

Comparison of (9) and (10) gives rise to

Theorem 1: If $f'\Phi \gtreqless kf\varphi$ for all $x \in (0,\bar{x})$, then DLR $\gtreqless k\text{ATC}$.

Consider for example a long, narrow city $(\Phi(x) = c_0 x)$ in which due to traffic congestion $f(x) = c_1 x^{.8}$. Then k=.8 and DLR =.8 ATC.

AS also demonstrated that Theorem 1 can be restated by defining the *transport cost shape* of the city *Ω(f)* to be the residential land area within a transport cost distance *f* of the CBD. Then since $\Omega(f(x))=\Phi(x)$, $\Omega' f' = \varphi$, so that $\frac{f'\Phi}{f\varphi} = \frac{f'\Omega}{f\Omega f'} = \frac{\Omega}{f\Omega'}$ and Theorem 1 can be restated as:

Theorem 1′: If $\frac{\Omega}{f\Omega'} \gtreqless k$ for all $f \in (0, \bar{f})$ then DLR $\gtreqless k$ ATC.

AS also showed that the result extends to multiple household groups who differ in their transport cost functions and their choices of lot size as a function of location. Index groups by *i* and residential rings (indexed away from the city center) by *j*, and let *i(j)* denote the group living in ring *j*. Then, where ρ is the number of rings:

$$\text{DLR} = \sum_{j=1}^{\rho} \int_{x_j}^{x_{j+1}} (R(x) - R_A)\varphi(x)dx$$

$$= -\sum_{j=1}^{\rho} \int_{x_j}^{x_{j+1}} R'(x)\Phi(x)dx$$

$$= \sum_{j=1}^{\rho} \int_{x_j}^{x_{j+1}} \frac{f'_{i(j)}(x)}{T_{i(j)}(x)}\Phi(x)dx \qquad (9')$$

and

$$\text{ATC} = \sum_{j=1}^{\rho} \int_{x_j}^{x_{j+1}} \frac{f_{i(j)}(x)}{T_{i(j)}(x)}\varphi(x)dx. \qquad (10')$$

Then the earlier results generalize when all groups have the same elasticity of transport costs with respect to distance from the CBD. Otherwise, the relationship between ATC and DLR can be bounded by the maximum and minimum elasticities across groups.

Several comments are in order:

1. All the results are essentially *capitalization* results[1] since the Muth rule, which is central to the derivation, indicates how marginal transport costs are "capitalized" into marginal land rent.

2. The results apply to any non-monocentric city with a competitive land markets in which differences in land rents are determined only by differences in accessibility for residents. Let a denote accessibility, $\hat{f}(a)$ transport costs as a function of accessibility and $\hat{\Phi}(a)$ the area of land with accessibility greater than a. Then all of the above analysis goes through, except for the obvious changes that arise due to rent *increasing* with accessibility.

But in interesting non-monocentric cities, land rents are determined not only by the accessibility for residents, but also by the productivity of different locations from the perspective of firms. The next section considers how the results are modified when these complications are treated.

3. The results apply to any city with a competitive land market in which differences in land rents are determined only by differences in residential accessibility. What is essential is that the Muth rule apply. Thus, the results apply with zoning, height restrictions, and/or traffic congestion. They do not apply, or are at least modified, if there are Ricardian differences in land (microclimate, the view, proximity to the ocean), if there is spatially-varying pollution, or if the level of taxes or public services vary over space. They do apply to any optimal (as opposed to equilibrium) city in which land rents depend only on accessibility, except of course that DLR is *shadow* differential land rents.

To see how the results extend when residential land rents capture not only differences in accessibility, consider a monocentric city in which pollution varies over space. To start, suppose that all residents are identical, that pollution is radially symmetric, with $K(x)$ the pollution concentration at x and $K'(x)<0$. Then a resident's locational choice problem is

[1] "capitalization" is an abuse of terminology since properly a future stream of benefits is capitalized into an asset value, which applies only in dynamic contexts.

$$\max_x U(C,T,K(x)) \text{ s.t. } \quad Y = C + R(x)T + f(x)$$

$$\Leftrightarrow \max_x U(Y - R(x)T - f(x), T, K(x)).$$

The first-order condition with respect to *x* is

$$U_C(-R'T - f') + U_K K' = 0, \qquad (11)$$

so that the generalization of the Muth rule is

$$R'T = -f' + \frac{U_K}{U_C}K'. \qquad (11')$$

Then[2]

$$\text{DLR} = -\int_0^{\bar{x}} R'(x)\Phi(x)dx$$

$$= \int_0^{\bar{x}} \frac{f'(x)}{T(x)}\Phi(x)dx - \int_0^{\bar{x}} \left(\frac{U_K}{U_C}\ \frac{K'}{T}\right)_x \Phi(x)dx, \qquad (12)$$

while (10) continues to apply. $-\frac{U_K}{U_C}\frac{K'}{T}$ is the rental discount a resident requires to live a unit distance closer to the CBD due to increased pollution. Thus, $-\int_0^{\bar{x}}\left(\frac{U_K}{U_C}\frac{K'}{T}\right)_x \Phi(x)dx$ can be referred to the capitalized pollution discount (CP). Likewise, $\int_0^{\bar{x}}\frac{f'(x)}{T(x)}\Phi(x)dx$ can be referred to as the capitalized transport cost premium (CT), so that (12) can be rewritten as

$$\text{DLR} = CT + CK. \qquad (12')$$

4. In all the analysis thus far, it has been possible to index locations uni-dimensionally, either because of radial symmetry or because locations can be unambiguously ranked in terms of accessibility or more generally desirability. But this is not always the case; for example, when different groups attach different relative valuations to improved accessibility and reduced pollution concentration, locations must be described using two coordinates. To

introduce the analytics, two-dimensional analysis is applied to a problem that can also be solved using one-dimensional analysis — the relationship between DLR and ATC in a city on a homogenous plane with a Manhattan network in which travel per unit distance in the north-south direction is t_0 and in the east-west direction t_1.

The one-dimensional analysis proceeds as follows. Let $f = t_0|y| + t_1|x|$ denote transport cost. The residential area with transport cost less than *f* (the transport cost shape of the city) is

$$\Omega(f) = 4 \int_0^{\frac{f}{t_0}} \int_0^{\frac{f - t_0 y}{t_1}} dxdy$$

$$= 4 \int_0^{\frac{f}{t_0}} \frac{f - t_0 y}{t_1} \; dy = \frac{2f^2}{t_1 t_0}.$$

From Theorem 1′, ATC=2DLR.

Consider now the two-dimensional analysis. Rent and transport cost are now functions of both *x* and *y*. Since the four quadrants of the city are symmetric, the relationship between ATC and DLR can be derived by examining only quadrant I, where both *x* and *y* are positive. The generalization of the Muth model is

$$R_x(x,y) = -\frac{t_1}{T(x,y)} \text{ and } R_y(x,y) = -\frac{t_0}{T(x,y)}. \qquad (13)$$

Also,

$$DLR = \int_0^{\frac{\bar{f}}{t_0}} \int_0^{\frac{\bar{f} - t_0 y}{t_1}} \left(R(x,y) - R_A\right) \; dxdy, \qquad (14)$$

where $\bar{f}$ is the transport cost to the boundary of the city at which $R(x,y)=R_A$. Integrate (14) by parts:

[2] With pollution it is possible that the residential bid-rent curve falls below R_A at locations inside the city. These locations are then farmed. This complication can be dealt with by defining $\Phi(x)$ to be the *equilibrium* residential area within *x* of the CBD.

$$\text{DLR} = \int_0^{\frac{\bar{f}}{t_0}} \left\{ \left(R(x,y)x \right)_0^{\frac{\bar{f}-t_0 y}{t_1}} - \int_0^{\frac{\bar{f}-t_0 y}{t_1}} R_x(x,y)x\, dx - R_A \left(\frac{\bar{f}-t_0 y}{t_1} \right) \right\} dy$$

$$= \int_0^{\frac{\bar{f}}{t_0}} \int_0^{\frac{\bar{f}-t_0 y}{t_1}} \frac{t_1 x}{T(x,y)}\, dxdy. \qquad (15a)$$

Reversing the order of integration:

$$\text{DLR} = \int_0^{\frac{\bar{f}}{t_1}} \int_0^{\frac{\bar{f}-t_1 x}{t_0}} \frac{t_0 y}{T(x,y)} dydx = \int_0^{\frac{\bar{f}}{t_0}} \int_0^{\frac{\bar{f}-t_0 y}{t_1}} \frac{t_1 x}{T(x,y)} dxdy \qquad (15b)$$

Then

$$\text{ATC} = \int_0^{\frac{\bar{f}}{t_0}} \int_0^{\frac{\bar{f}-t_0 y}{t_1}} \frac{t_0 y + t_1 x}{T(x,y)} dxdy \quad = 2\text{DLR}. \qquad (16)$$

Now consider the same city, except that there are two groups in the population which have different transport cost functions $f^1(x,y)$ and $f^2(x,y)$. The groups may be differentiated by income or modal choice. In general, a two-dimensional analysis is needed to treat this problem. Let $I^i(x,y)$ be an indicator function, equaling 1 if group *I* lives at *(x,y)* and zero otherwise, for *i*=1,2. Then

$$\text{DLR} = \sum_{i=1}^{2} \iint_A I^i(x,y)(R(x,y) - R_A) dxdy, \qquad (17)$$

where A denotes the residential area of the city. The integration by parts yields

$$\text{DLR} = \sum_{i=1}^{2} \iint_A I^i(x,y) R_x x \; dxdy$$

$$= \sum_{i=1}^{2} \iint_A \frac{I^i(x,y) f_x^{\,i}(x,y)x}{T^i(x,y)} dx \; dy = \sum_{i=1}^{2} \iint_{A\;i} \frac{I^i(x,y) f_y^{\,i}(x,y)y}{T^i(x,y)} dx \; dy \qquad (18)$$

Also

$$\text{ATC} = \sum_{i=1}^{2} \iint_A \frac{I^i(x,y) f^i(x,y)}{T^i(x,y)} dxdy. \qquad (19)$$

Eqs. (18) and (19) give an implicit relationship between DLR and ATC. If the two groups have different linear transport cost functions, it is still the case that ATC=2DLR.

Thus, the results do extend to situations where a two-dimensional analysis is needed.

5. How do the results extend when there is housing? Consider first the case where housing is non-durable. The relationship between ATC and DLR remains unchanged. Residents simply combine some of their composite good with their land to produce housing. What about the relationship between differential housing rents (DHR) and aggregate transport costs? When households derive utility from just housing and other goods, the analog to the Muth condition, for the case of linear transport costs, is

$$R_H'(x) = \frac{-t}{H(x)}, \tag{20}$$

where $R_H(x)$ is the *housing* rent function and $H(x)$ the quantity of housing consumed at x.

$$\text{DHR} = \int_0^{\bar{x}} \left(R_H(x) - R_H(\bar{x})\right)\Phi_H(x)dx, \tag{21}$$

where $\Phi_H(x)dx$ is the floor area of *housing* between x and $x+dx$. Integration by parts gives

$$\begin{aligned} \text{DHR} &= -\int_0^{\bar{x}} R_H'(x)x\Phi_H(x)dx \\ &= \int_0^{\bar{x}} \frac{t}{H(x)}\Phi_H(x)\ \ dx. \end{aligned} \tag{22}$$

Also

$$\text{ATC} = \int_0^{\bar{x}} \frac{tx}{H(x)}\varphi_H(x)\ \ dx. \qquad \left(\varphi_H(x) = \frac{d\Phi_H(x)}{dx}\right) \tag{23}$$

Thus, there is a completely analogous relationship between DHR and ATC as between DLR and ATC. Once an immobile structure is built on a piece of land, the market land rent there is not defined since it is not possible to rent the land without the structure on it. Thus, with durable housing, DLR is not well defined. The relationship between DHR and ATC does, however, continue to apply with durable housing since *housing* rent remains well-defined.

6. The analysis thus far has treated trip demand as completely inelastic. Suppose instead that the number of trips taken at *x* depends negatively on the location's accessibility: $\eta(x) = \hat{\eta}(f(x))$, $\eta' < 0$. Then the Muth rule becomes

$$R'(x) = \frac{-\eta(x) f'(x)}{T(x)}, \tag{24a}$$

and so

$$\begin{aligned} \text{DLR} &= -\int_0^{\bar{x}} R'(x)\Phi(x)dx \\ &= \int_0^{\bar{x}} \frac{\eta(x) f'(x)}{T(x)}\Phi(x)dx \end{aligned} \tag{24b}$$

and

$$\text{ATC} = \int_0^{\bar{x}} \frac{\eta(x) f(x)}{T(x)}\varphi(x)dx. \tag{24c}$$

Thus, the relationship between ATC and DLR is unaffected by elastic trip demand, except that *f(x)* is interpreted as per-trip transport costs at *x*.

3. Analogous Relationships in Non-monocentric Cities

We have already seen that our analysis extends to non-monocentric cities when all the urban land area is devoted to residential land/housing. With identical households, "distance to the CBD" is replaced by "accessibility", and the analysis goes through as before. With heterogeneous households, a two-dimensional analysis is in general necessary, and (18) and (19) apply. The difficult aspect of the extension to non-monocentric cities is the treatment of non-residential land use, which we shall term generically commercial land use.

Consider the location decision of a firm. In making this decision, it takes into account not only the accessibility of alternative locations, but also how wages and productivity vary over location. Thus, not only differences in transport costs, but also differences in wages and productivity are capitalized into land rents. The economics of the process was first considered by Fujita and Ogawa (1981), then by Roback (1982), and subsequently by

Sivitanidou and Wheaton (1992). The analysis below will draw heavily on the insights provided by those papers and apply them to examine the relationship between urban economic aggregates in cities in which there is both commercial and residential land use.

To fix ideas, consider what Fujita and Ogawa term a "completely mixed urban configuration" in which firms and households are completely interspersed, with each household living right next to the firm where it works so that commuting costs are zero. Firms benefit from the proximity of other firms, so that firms that are more centrally located are more productive. This translates into higher rents at more central locations. The land market is competitive. Consequently workers at more central locations have to pay higher rents, and require higher wages to compensate them for the higher rent. What is the relationship between urban economic aggregates in this city?

To simplify the analysis, the following assumptions are made: i) the city is long and narrow (of unit width), which permits one-dimensional analysis; ii) there are spatially-attenuating Marshallian economies of scale, with each firm perceiving that it operates under constant returns to scale, so that firm size is indeterminate; iii) nevertheless, for analytical convenience, each firm is taken to occupy a unit area of land. x is measured so that $x=0$ corresponds to the city center.

Following Fujita and Ogawa, a *location potential function* is defined which measures the productivity of a location as a function of proximity to other firms. To simplify, it is assumed that the function $P(x)$ has the general form

$$P(x) = \int_{-\bar{x}}^{\bar{x}} n(y)D(|x-y|)dy, \tag{25}$$

where $n(y)$ is the number of workers at y and $D(|x-y|)$ is the distance-decay function, and that output per unit area of commercial land at y is

$$q(x) = P(x)g(n(x)), \qquad g' > 0, g'' < 0. \tag{26}$$

Let $h(x)$ be the proportion of land at x used by households and $b(x)$ be the proportion used by business.

Now consider a representative firm's profit function:

$$\Pi(x) = q(x) - w(x)n(x) - R_b(x)$$
$$= P(x)g(n(x)) - w(x)n(x) - R_b(x), \qquad (27)$$

where *w(x)* is the wage rate at *x* and $R_b(x)$ is the business bid rent. By assumption there is a completely mixed urban configuration in equilibrium. This implies that production must be equally profitable at all urban locations. Furthermore, free entry and exit of firms is assumed, which implies that equilibrium profits are zero. Then:

$$n(x): \qquad Pg_n - w = 0 \qquad (28a)$$

$$x: \qquad P'g - w'n - R_b' + \left(Pg_n - w\right)n' = 0 \qquad (28b)$$

$$\Pi = 0: \qquad R_b = Pg - wn. \qquad (28c)$$

Eq. (28a) is the labor-demand condition, (28b) the equal-profit-at-all-urban-locations condition, and (28c) the zero-profit condition.

The representative worker's utility-maximization problem is

$$\max_{x,T,C} \; U(C,T) \qquad s.t. \quad I + w(x) - R_h(x)T - C = 0 \qquad (29)$$

where *I* is lump-sum income and since $R_h(x)$ is the residential bid rent at *x*. Since by assumption there is a completely mixed urban configuration, there are workers living at all urban locations. Then

$$T: \qquad -U_C R_h + U_T = 0 \qquad (30a)$$

$$\text{x}: \qquad U_C\left(w' - R_h' T\right) = 0 \qquad (30b)$$

Eq. (30a) is lot-size-choice condition, and (30b) the equal-utilities-at-all-urban-locations condition.

There are three other equilibrium conditions:

$$n(x)b(x) - \frac{h(x)}{T(x)} = 0 \;\; \text{(with } b(x) = 1 - h(x)\text{)} \quad \forall x \in \boldsymbol{U} \qquad (31a)$$

$$R^b(x) - R(x) = 0 \qquad \forall x \in \boldsymbol{U} \qquad (31b)$$

$$R(-\bar{x}) = R(\bar{x}) = R_A, \qquad (31c)$$

where U is the set of the urban locations. Eq. (31a) indicates that at all urban locations, all land is used and the number of workers employed there equals the number of workers living there; (31b), that equilibrium in the land market requires that the residential bid rent equal the commercial bid rent at all urban locations; and (31c) that at the urban boundary the urban rent equals the agricultural rent.

Now

$$\begin{aligned}
\text{DLR} &= \int_{-\bar{x}}^{\bar{x}} \big(R(x) - R_A\big)dx = 2\int_0^{\bar{x}} \big(R(x) - R_A\big)dx \quad \text{(using (31b))} \\
&= -2\int_0^{\bar{x}} R_0'(x)xdx \quad \text{(integration by parts and (31c))} \\
&= -2\left\{\int_0^{\bar{x}} R'(x)b(x)xdx + \int_0^{\bar{x}} R'(x)h(x)xdx\right\} \\
&= -2\left\{\int_0^{\bar{x}} (P'g - w'n)bxdx + \int_0^{\bar{x}} \frac{w'}{T}hxdx\right\} \quad \text{(using (28b) and (30b))} \\
&= -2\int_0^{\bar{x}} P'gbxdx\,. \quad \text{(using (31a))} \qquad (32a)
\end{aligned}$$

The terms on the RHS is the *capitalized agglomeration premium* (CG), and is analogous to the capitalized transport cost premium and the capitalized pollution discount introduced earlier. Thus, (32) may be written succinctly as

$$\text{DLR} = \text{CG}. \qquad (32b)$$

Since there is no commuting, aggregate transport costs are of course zero.

There are still a few loose ends with respect to the characterization of equilibrium.

1. The disposition of land rents affects the equilibrium but not the relationship between aggregates derived in the paper. For the sake of concreteness, however, assume that land rents go to absentee landlords.

2. How are the population and/or the utility level of workers determined? To begin, take as fixed the utility level. Given the utility level and the agricultural rent, the equilibrium wage and lot size at the boundary locations, $w(\bar{x})$ and $T(\bar{x})$ respectively, can be solved for. From (28a) and (28c), $P(\bar{x})$ and $n(\bar{x})$ can be solved. Then from (31a) $b(\bar{x})$ can be solved for.

Now conjecture an equilibrium function $n^{(0)}(x)$ which from (25) yields $P^{(0)}(x)$ and $P'^{(0)}(x)$. From (28b) and (30b), $w'(\bar{x})$ and $R'(\bar{x})$ can be calculated, from which $w(\bar{x} - dx)$ and $R(\bar{x} - dx)$ can be calculated, etc. Proceeding recursively to smaller and smaller values of *x* and stopping at $x = 0$, yields $n^{(1)}(x)$. A function $\tilde{n}(x)$ which corresponds to a fixed point of this procedure is a candidate equilibrium.

3. A final equilibrium condition is that the allocation corresponding to a candidate equilibrium $n(x)$ is consistent with a completely mixed urban configuration being the equilibrium configuration. A necessary and sufficient candidate for this is that no worker have an incentive to work at a location different from where he resides. This is equivalent to the condition that $|w'(x)| \leq t$ at all urban locations, and will clearly be satisfied for sufficiently large *t*.

Uniqueness of *n(x)* conditional on existence remains to be established.

Now consider how locational productivity differentials are capitalized into wages and rents in such an equilibrium. From (28b)

$$P'g = w'n + R'_b$$

$$= w'n + R' = w'n + \frac{w'}{T} \qquad \text{(using (30b)}$$

$$= w'\left(n + \frac{nb}{1-b}\right) \qquad \text{(using (31a))}$$

$$= w'n\left(\frac{1}{1-b}\right) \tag{34a}$$

and $$P'g = w'n + R' = R'(nT + 1) \qquad \text{(using (30b)}$$

$$= R'\left(\frac{1}{b}\right). \qquad \text{(using (31a))} \tag{34b}$$

Thus, the extent to which productivity differences are capitalized into rents compared to wages depends on *b*, the proportion of land at that location in commercial use. From (34a) and (34b):

$$w'n = (1-b)P'g \text{ and } R' = bP'g. \tag{35}$$

Define

$$\varnothing = \int_{-\bar{x}}^{\bar{x}} Pgbdx = 2\int_0^{\bar{x}} Pgbdx \tag{33a}$$

to be gross urban output. Comparing (32a) and (33), and making use of (32b), yields

$$\frac{\text{DLR}}{\varnothing} = \frac{-\int_0^{\bar{x}} P'gbxdx}{\int_0^{\bar{x}} Pgbdx} \tag{33b}$$

which provides bounds on the ratio of DLR to gross output; in particular $\bar{\varepsilon}\varnothing > \text{DLR} > \underline{\varepsilon}\varnothing$, when $\bar{\varepsilon} = \max_{x\in(0,\bar{x})} \varepsilon$, $\underline{\varepsilon} = \min_{x\in(0,\bar{x})} \varepsilon$, and $\varepsilon = \frac{P'x}{P}$ is the elasticity of productivity with respect to distance from the city center.

The above analysis assumed that there are no restrictions on commercial and residential density. Restrictions on residential density do not affect the analysis — (30b) still applies. Neither do restrictions on commercial density — as characterized[1] by *n(x)*. Assume a restriction on maximum labor density, $\bar{n}$. Consider the term $(Pg_n - w)n' = 0$ in (28b); at those locations where the density restriction binds, $n' = 0$, and at other locations $Pg_n - w = 0$. If, however, the density restriction varies over space, then

$$\begin{aligned} \text{DLR} &= -2\left\{\int_0^{\bar{x}} P'gbdx - 2\int_0^{\bar{x}} \left(Pg_n(\bar{n}(x)) - w\right)\bar{n}'(x)bdx\right\} \\ &= CG + EB, \end{aligned} \tag{33c}$$

where

$$\text{EB} = \int_{-\tilde{x}}^{\tilde{x}} \left(Pg_n(\bar{n}(x)) - w\right)\bar{n}'(x)bdx \tag{33d}$$

[1] We may imagine that a fixed floor area is required for each worker. *g(n)* is then the net product after subtracting off the resources used to provide the floor area needed for *n* workers.

is the excess burden associated with the spatially varying component of the density restriction.

Let us now turn to equilibrium configurations in which residential and commercial land are everywhere separated. Figure 2 displays a possible equilibrium land use pattern.

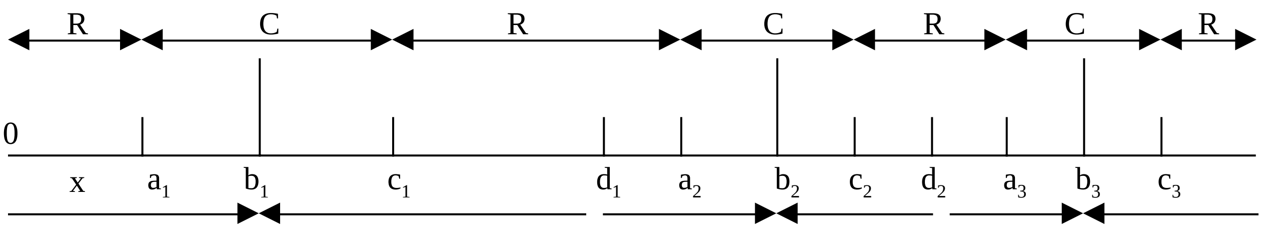

Figure 2

Locations are measured from the left to the right, with x=0 denoting the left boundary of the city. *R* denotes residential area and *C* commercial area. Commercial and residential districts are indexed from left to right. Define a_i to be the left edge of a commercial zone and c_1 the right edge.

It is well-known and can be quite easily established by contradiction that no worker travels all the way across a subcenter. Thus, all the workers living to the west of a_1 work in the commercial district between a_1 and c_1 etc. Furthermore, there is no cross commuting in equilibrium. This implies that there is a location in the interior of commercial district i, b_i , such that at all commercial locations in the district to the left of it workers travel to there from left to right, and at all commercial locations in the district to the right of it, workers travel from right to left. This also implies that there is a location in the interior of each interior residential district d_i such that at all residential locations to the left of it workers travel to the commercial district to their left, and at all residential locations to the right of it workers travel to the commercial district to their right. The resulting commuting directions are indicated by the arrows at the bottom of the figure. These equilibrium conditions considerably simplify the analysis. Then

$$\text{DLR} = \int_0^{a_1} \left(R(x) - R_A\right)dx + \sum_{i=1}^{k}\left\{\int_{a_i}^{b_i}\left(R_b(x) - R_A\right)dx + \int_{b_i}^{c_i}\left(R_b(x) - R_A\right)dx\right\}$$

$$+\sum_{i=1}^{k-1}\left\{\int_{c_i}^{d_i}\left(R(x) - R_A\right)dx\right\} + \sum_{i=1}^{k-1}\left\{\int_{d_i}^{a_{i+1}}\left(R(x) - R_A\right)dx\right\} + \int_{c_k}^{\bar{x}}\left(R(x) - R_A\right)dx. \quad (36)$$

Now

$$\int_0^{a_1}\left(R(x) - R_A\right)dx = \left(R(x)x\right)_0^{a_1} - \int_0^{a_1} R'(x)xdx - R_A a_1$$

$$= R(a_1)a_1 - \int_0^{a_1}\frac{tx}{T(x)}dx - R_A a_1. \quad (37)$$

Also,

$$\int_{a_i}^{b_i}\left(R_b(x) - R_A\right)dx = \left(R_b(x)x\right)_{a_i}^{b_i} - \int_{a_i}^{b_i}R_b'(x)xdx - R_A(b_i - a_i)$$

$$= R_b(b_i)b_i - R_b(a_i)a_i - \int_{a_i}^{b_i}P'gxdx + \int_{a_i}^{b_i}w'nxdx - R_A(b_i - a_i).$$

Between a_i and b_i, the wage has to compensate for the increase in transport cost: $w'=t$.

Thus,

$$\int_{a_i}^{b_i}\left(R_b(x) - R_A\right)dx = R_b(b_i)b_i - R_b(a_i)a_i - \int_{a_i}^{b_i}P'gxdx + \int_{a_i}^{b_i}tnxdx - R_A(b_i - a_i). \quad (37b)$$

Between b_i and c_i, $w'=-t$. Thus:

$$\int_{b_i}^{c_i}\left(R_b(x) - R_A\right)dx = R_b(c_i)c_i - R_b(b_i)b_i - \int_{b_i}^{c_i}P'gxdx - \int_{b_i}^{c_i}tnxdx \quad (37c)$$

$$\int_{c_i}^{d_i}\left(R(x) - R_A\right)dx = R(d_i)d_i - R(c_i)c_i + \int_{c_i}^{d_i}\frac{tx}{T(x)}dx - R_A(d_i - c_i) \quad (37d)$$

$$\int_{d_i}^{a_{i+1}}\left(R(x) - R_A\right)dx = R(a_{i+1})a_{i+1} - R(d_i)d_i - \int_{d_i}^{a_{i+1}}\frac{tx}{T(x)}dx - R_A(a_{i+1} - d_i) \quad (37e)$$

$$\int_{c_k}^{\bar{x}}\left(R(x) - R_A\right)dx = R(\bar{x})\bar{x} - R(c_k)c_k + \int_{c_k}^{\bar{x}}\frac{tx}{T(x)}dx - R_A(\bar{x} - c_k). \quad (37f)$$

Hence,

$$\text{DLR} = -\int_0^{a_1}\frac{tx}{T(x)}dx + \sum_{i=1}^{k}\left\{\int_{a_i}^{b_i}\left(tn - P'g\right)xdx - \int_{b_i}^{c_i}\left(tn + P'g\right)xdx\right\}$$

$$+\sum_{i=1}^{b-1}\left\{\int_{c_i}^{d_i}\frac{tx}{T(x)}dx-\int_{d_i}^{a_i+1}\frac{tx}{T(x)}dx\right\}+\int_{c_k}^{\bar{x}}\frac{tx}{T(x)}dx\,. \tag{38}$$

Now,

$$\text{ATC } =\int_0^{a_1}\frac{t(a_1-x)}{T(x)}dx+\sum_{i=1}^{k}\left\{\int_{a_i}^{b_i}t(x-a_i)ndx+\int_{b_i}^{c_i}t(c_i-x)ndx\right\}$$

$$+\sum_{i=1}^{k-1}\left\{\int_{c_i}^{d_i}\frac{t(x-c_i)}{T(x)}dx+\int_{d_i}^{a_i+1}\frac{t(a_{i+1}-x)}{T(x)}dx\right\}+\int_{c_k}^{\bar{x}}\frac{t(x-c_k)}{T(x)}dx \tag{39}$$

$$=-\int_c^{a_1}\frac{tx}{T(x)}+\sum_{i=1}^{k}\left\{\int_{a_i}^{b_i}tnxdx-\int_{b_i}^{c_i}tnxdx\right\}$$

$$+\sum_{i=1}^{k-1}\left\{\int_{c_i}^{d_i}\frac{tx}{T(x)}dx-\int_{d_i}^{a_{i+1}}\frac{tx}{T(x)}dx\right\}+\int_{c_k}^{\bar{x}}\frac{tx}{T(x)}dx$$

$$+\int_0^{a_1}\frac{a_1t}{T(x)}dx+\sum_{i=1}^{k}\left\{-\int_{a_i}^{a_i}tndx+\int_{b_i}^{ci}c_itndx\right\}$$

$$+\sum_{i=1}^{k-1}\left\{-\int_{c_i}^{d_i}\frac{c_it}{T(x)}+\int_{d_i}^{a_i+1}\frac{a_{i+1}t}{T(x)}dx\right\}-\int_{c_k}^{\bar{x}}\frac{c_kt}{T(x)}dx\,. \tag{39′}$$

Now, $\int_0^{a_1}\frac{1}{T(x)}dx=\int_{b_1}^{c_1}ndx$, etc. Substituting these relationships into (39′) yields

$$\text{ATC } =-\int_0^{a_1}\frac{tx}{T(x)}dx+\sum_{i=1}^{k}\left\{\int_{a_i}^{b_i}tnxdx-\int_{b_i}^{c_i}tnxdx\right\}$$

$$+\sum_{i=1}^{k-1}\left\{\int_{c_i}^{d_i}\frac{tx}{T(x)}dx-\int_{d_i}^{a_{i+1}}\frac{tx}{T(x)}dx\right\}+\int_{c_k}^{\bar{x}}\frac{tx}{T(x)}dx\,. \tag{40}$$

Comparing (38) and (40) yields

$$\text{DLR } +\sum_{i+1}^{k}\left\{\int_{a_i}^{b_i}P'gxdx+\int_{b_i}^{c_i}P'gxdx\right\}=ATC\,. \tag{41}$$

Now, analogous to the case with the completely mixed urban configuration, we define the capitalized agglomeration premium to be

$$\text{CG } =-\sum_{i=1}^{k}\left\{\int_{a_i}^{c_i}P'gxdx\right\}. \tag{42}$$

Substituting (41) into (40) yields

$$\text{DLR=CG + ATC.} \tag{43}$$

We have already seen how density restrictions affect the analysis. What about restrictions on land use? Suppose for example that commercial area 1 is restricted in size such that $R_b(a_1) > R(a_1)$ and $R_b(c_1) > R(c_1)$. Then from (37a), (37b), and (37c)

$$\text{DLR} = \text{CG} + \text{ATC} - (R_b(a_1) - R(a_1))a_1 + (R_b(c_1) - R(c_1))c_1 . \tag{44a}$$

In general,

$$\text{DLR} = \text{CG} + \text{ATC} - \text{DWL}, \tag{44b}$$

where DWL is the deadweight loss due to land use restrictions.

Now let us turn to capitalization results. Consider two locations e_1 and e_2 in adjacent commercial areas 1 and 2:

$$R(e_2) - R(e_1) = \int_{e_1}^{c_1} R_b' dx + \int_{e_1}^{a_2} R' dx + \int_{a_2}^{e_2} R_b' dx$$

$$= \int_{a_1}^{c_1} (P'g - w'n) dx + \int_{C_1}^{d_1} \frac{t}{T} dx + \int_{d_1}^{a_2} -\frac{t}{T} dx + \int_{a_2}^{e_2} (P'g - w'n) dx$$

Extensions — To Be Completed

i. Different groups — jobs, tastes, transport costs

ii. Different industries

iii. Two dimensions

iv. Mixed and separated areas

v. Extend to congestion

vi. Capitalization into wages

vii. Explain CD

viii. Capitalization into rent

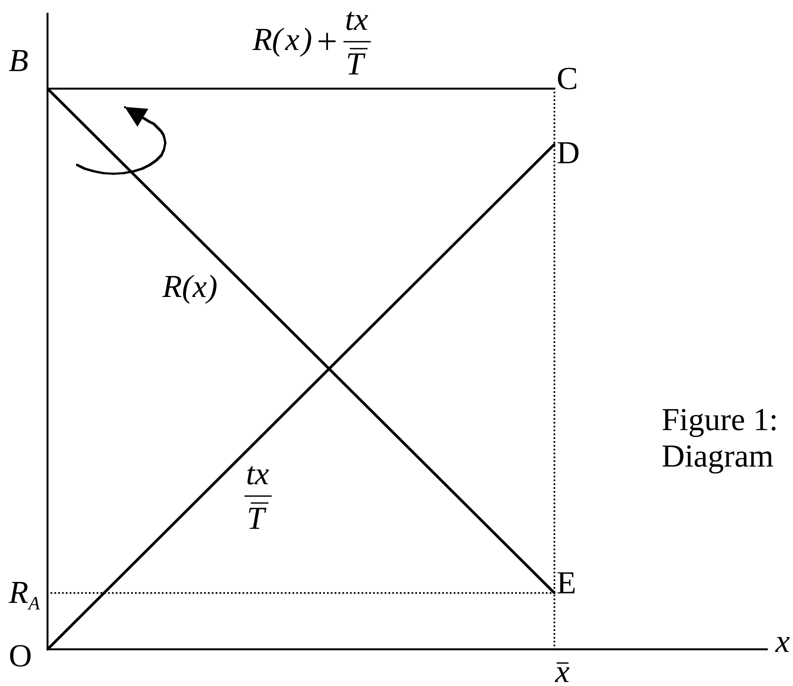

Figure 1: The Mohring Diagram