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Abstract

Shape invariance is a property of demand functions that is convenient for semiparametric demand modelling. All known shape invariant demands are derived from utility functions that, up to monotonic transformation, are called IB/ESE (independent of base - equivalence scale exact) utility functions, because they yield IB/ESE equivalence scales, which are widely used in welfare calculations. This paper provides a counterexample, i.e., a shape invariant demand system that is *not* derived from a transform of IB/ESE utility. A general theorem is then provided that characterizes all shape invariant demand systems. The usual practice of equating shape invariance with the IB/ESE utility class is shown to be not quite right, but it can be made valid by testing for the small class of exceptions noted here. In particular, all the exceptions have rank two, so any rank three or higher shape invariant system must be derived from transforms of IB/ESE utility.

Keywords: Shape Invariance, Equivalence Scales, Engel curves, Consumer demand, Demand Systems, Utility, Cost Functions.

JEL Codes: D11, D12, C31, C51
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1 Introduction

Shape invariance is a property of demand functions that is convenient for semiparametric demand modelling, and has been found to at least approximately hold empirically in some data sets. Shape invariant demand functions are known to arise from the class of utility functions that, up to monotonic transformation, are called IB/ESE (independence of base - equivalence scale exactness) utility functions. These utility functions yield IB/ESE equivalence scales, which have many important implications for social welfare calculations and are widely used in practice. It has been generally assumed that all shape invariant demand functions are derived from this class.

This paper first provides a counterexample. A shape invariant demand system is given that is *not* derived from a transform of IB/ESE utility. A general theorem then shows that all shape invariant demand systems are derived either from transforms of IB/ESE utility or from a restrictive class of exceptions (including the provided counterexample). Thus the usual practice of equating shape invariance with transformations of IB/ESE utility is not quite right, but it can be made valid by testing for the class of exceptions noted here.

Some simple sufficient conditions on Engel curves are then provided that can be applied empirically to test for these exceptional cases. For example, it is shown that all the exceptions have demand rank two, so all demand systems that have shape invariance and rank higher than two must be derived from transforms of IB/ESE utility.

2 Definitions

Let y denote the logarithm of the total dollars a consumer spends on goods, and for goods $i = 1, ..., n$, let w_i denote the consumer's budget share for good i, that is, the fraction of total dollar expenditures $\exp(y)$ that the consumer devotes to purchases of good i. Let A denote a vector of attributes (fixed or random utility parameters) of the consumer that affect his tastes. Let p denote the n vector of logged prices $(p_1, ..., p_n)$ the consumer faces for goods. Assume that the consumer determines $w = (w_1, ..., w_n)$ to maximize utility given a standard budget constraint. Specifically, Assumption A1 is assumed to hold.

ASSUMPTION A1: A consumer has an indirect utility function $u = U(p, y, A)$ that possesses the following standard regularity properties: Nonincreasing, twice differentiable, and quasiconvex in p, strictly increasing and differentiable in y, and homogeneous of degree zero in (e^p, e^y). The consumer's budget share Marshallian demand functions are given by Roys identity

$$w_i = -\frac{\partial U(p, y, A)/\partial p_i}{\partial U(p, y, A)/\partial y}$$

Engel curves are defined as the functions ω_i such that, in a given price regime, $w_i = \omega_i(y, A)$. Engel curves are often highly nonlinear in y, but not necessarily in A, and the vector of attributes A is often long with many discrete elements, so for dimension reduction it would be natural to specify a partly linear form $w_i = A'b_i + G_i(y)$, or more generally an additive model $w_i = H_i(A) + G_i(y)$ where H_i may be relatively tightly parameterized and G_i is semiparametrically or nonparametrically estimated. The difficulty with these specifications is that, except for very specific choices of the functions G_i, these models are inconsistent with utility maximization.

General solutions to this problem, proposed by Blundell, Duncan and Pendakur (1998), are shape invariant Engel curves, which are Engel curves of the form

$$w_i = H_i(A) + G_i\left[y - B(A)\right]. \tag{1}$$

See also Pendakur (1998), who provides a general nonparametric test for shape invariance, and Blundell, Chen, and Kristensen (2003) who implement semiparametric estimators of shape invariant Engel curves. Without reference to demand systems, nonparametric estimators of collections of shape invariant regressions were proposed by Hardle and Marron (1990) and Pinkse and Robinson (1995). Other tests and estimators of shape invariant Engel curves include Gozalo (1997), Yatchew, Sun, and Deri (2003), and Stengos, Sun, and Wang (2006). Many empirically popular specifications for demand systems possess shape invariant Engel curves. Examples include Jorgenson, Lau, and Stoker's (1982) Translog demand systems and Banks, Blundell, and Lewbel's (1997) Quadratic Almost Ideal Demand System.

Define shape invariant (Marshallian) demand functions as

$$w_i = H_i(p, A) + G_i\left(y - B(p, A), p\right) \tag{2}$$

which is just equation (1), where the log price regime vector p is now explicitly included in the functions H_i, G_i, and B where it was formerly implicit.

Up until the present, the only way shape invariant Engel curves have been shown to be consistent with utility maximization is as follows. Suppose consumer's have indirect utility functions of the form

$$u = V(y - B(p, A), p) \tag{3}$$

for some functions V and B. This is equivalent to a log cost (expenditure) function of the form $y = C(p, u, A) = \Psi(p, u) + B(p, A)$ for some functions Ψ and B, obtained by solving equation (3) for y. Since demand functions are unaffected by monotonic transformations of utility, we may more generally assume the indirect utility function is

$$u = T\left[V(y - B(p, A), p), A\right], \tag{4}$$

for any function T that is strictly monotonically increasing in its first argument, or equivalently a log cost function of the form

$$y = C(p, u, A) = \Psi\left[p, \tau(u, A) + B(p, A)\right]. \tag{5}$$

It follows immediately from Roy's identity that the Marshallian budget share demand functions arising from equation (3) or equation (4) are

$$w_i = \frac{\partial B(p, A)}{\partial p_i} - \frac{\partial V(y - B(p, A), p)/\partial p_i}{\partial V(y - B(p, A), p)/\partial y} \tag{6}$$

which is in the form of equation (2) and so is shape invariant.

As an example, consider indirect utility functions of the form

$$u = \sum_{i=1}^{n-1}(p_n - p_i)\Gamma_i\left[y - B(p, A)\right] \tag{7}$$

for some functions Γ_i, $i = 1, ..., n-1$. Applying Roy's identity to equation (7) yields shape invariant demand functions (2) where $G_i(x, p)$ is proportional to $\left[\partial\Gamma_i(x)/\partial x\right]^{-1}$ for all goods i except $i = n$, which has budget share demands given by one minus the sum of the other goods shares. The only constraint on the Γ_i functions and hence on the associated Engel curve components G_i are the inequalities implied by Assumption A1, which shows that shape invariant demand functions allow for almost unlimited G_i Engel functions. Some of the previously cited papers use unconstrained nonparametric regressions to estimate these functions.

A (logged) equivalence scale is defined as $C(p, u, A) - C(p, u, A_0)$ for some reference level of attributes A_0. An equivalence scale is defined to be IB (Independent of Base) or have ESE (Equivalence Scale Exactness) if the scale is independent of u. Consumers have IB/ESE equivalence scales if and only if they have indirect utility functions in the form of equation (3). See Lewbel (1997) for a survey of equivalence scales, and Blackorby and Donaldson (1993) and Lewbel (1989) for further definitions and details regarding IB/ESE scales. Many empirical applications of equivalence scales assume IB/ESE. A prominent example is Jorgenson and Slesnick (1993). While IB/ESE is sometimes empirically rejected, estimated equivalence scales that are not constrained to be IB/ESE are nevertheless often found to be numerically close to IB/ESE. See, e.g., Blundell and Lewbel (1991), Dickens, Fry, and Pashardes (1993), Pashardes (1995), Gozalo (1997), Pendakur (1998), Blundell, Duncan and Pendakur (1998), Lyssiotou, (2003), Yatchew, Sun, and Deri (2003), and Stengos, Sun, and Wang (2006).

It immediately follows from equations (2), (3), and (6) that if consumers have IB/ESE equivalence scales, then they must also have shape invariant Engel curves. Call equation (4) a TIB, for "Transformed IB," indirect utility function, since it is a monotonic transformation of any indirect utility function that generates IB equivalence scales. Given demand functions (6), the TIB utility function (4) is observationally equivalent to the indirect utility function (3) that generates IB scales, so in applications if demand functions are given by equation (6) for some functions V and B, then IB/ESE equivalence scales can be constructed by making the untestable assumption that utility is given by (3) rather than (4). This is, e.g., the method used by Jorgenson and Slesnick (1993).

The above results show that consumers that have TIB utility functions (4) must also have shape invariant demand functions, and hence shape invariant Engel curves. But is the converse true? No example of a shape invariant demand function that does not come from TIB utility currently exists in the literature. Also, if the converse were true, then tests of shape invariance such as Pendakur (1998) would equal tests of TIB utility and hence would be tests of all the observable implications of IB/ESE equivalence scales. However, the next section shows that the converse is not true.

3 A Shape Invariance Exception

It will now be shown, by means of a counterexample, that a consumer can have shape invariant demand functions without having a TIB utility function. Consider the indirect utility function

$$u = \left[\frac{1}{y - b(p, A) - m(A)} + \frac{1}{2m(A)} \right] e^{2m(A)s(p)} \tag{8}$$

for some scalar valued function $m(A)$, some function $b(p, A)$ such that $e^{b(p,A)}$ is linearly homogeneous in prices e^p, and some function $s(p)$ that is homogeneous of degree zero in prices e^p. These homogeneity restrictions are required by Assumption A1.

THEOREM 1. Let Assumption A1 hold and assume that a consumer's indirect utility function is equation (8). Then this consumer has shape invariant demand functions but does not have a TIB utility function.

Formal proofs are in the Appendix, but informally equation (8) does not look like equation (4) and so does not look like TIB, but by Roys identity the demand functions arising from equation (8) are

$$w_i = \left(\frac{\partial b(p, A)}{\partial p_i} - m(A)^2 \frac{\partial s(p)}{\partial p_i}\right) + \left(\left[y - b(p, A)\right]^2 \frac{\partial s(p)}{\partial p_i}\right) \tag{9}$$

which are in the form of equation (2) and hence are shape invariant, where the first term in parentheses is $H_i(p, A)$ and the second term in parentheses is $G_i\,(y - B(p, A), p)$.

This example also shows that it is possible to have nonlinear Engel curves that are additive in functions of A and y, since if we replace $b(p, A)$ equation (8) with $b(p) = b(p, A_0)$ for some constant A_0, then we obtain budget share demand functions (9) with $b(p)$ in place of $b(p, A)$, which yields Engel curves of the additive form

$$w_i = H_i(A) + G_i(y) = \left[b_i - m(A)^2 s_i\right] + \left[(y - b)^2 s_i\right]$$

for constants b, s_i, and b_i, $i = 1, ..., n$.

4 Characterizing Shape Invariance With and Without TIB

Theorem 2 below provides the necessary and sufficient conditions for a consumer to have shape invariant demand functions, and Corollary 1 describes the corresponding preferences in terms of cost functions.

THEOREM 2: Let Assumption A1 hold. A consumer has shape invariant demand functions if and only if, for every good i, the consumer either has budget share demand functions (6) obtained from the TIB indirect utility function (4), or the consumer has budget share demand functions of the form

$$w_i = \frac{\partial B(p, A)}{\partial p_i} + \frac{\partial S(p)}{\partial p_i}\left[h(S(p), A) + g\,(y - B(p, A), S(p))\right], \quad i = 1, ..., n \tag{10}$$

for some functions B, S, h, and g, where $g\,(y - B, S)$ is nonlinear in y and there exists some p, A for which $h\,(S\,(p)\,, A) \neq 0$.

Note that equation (6) demands always come from TIB utility, and all shape invariant demands are either equations (6) or (10), but it is possible for (10) to be either TIB or non-TIB, since it is possible for demand functions to be simultaneously in the form of equation (6) and (10). An example is when h is zero. The restrictions in Theorem 2 that g be nonlinear and h be nonzero are not necessary, but are included without loss of generality because when these restrictions do not hold, equation (10) demand functions are TIB.

COROLLARY 1: Let Assumption A1 hold. A consumer has shape invariant demand functions if and only if the consume either has TIB utility, given by the log cost function (5), or has utility given by the log cost function

$$y = C(p, u, A) = c\left[S(p), u, A\right] + B(p, A) \tag{11}$$

for some functions c, B, and S, where the function c is a solution to the differential equation

$$\frac{\partial c(s, u, A)}{\partial s} = h(s, A) + g\left[c(s, u, A), s\right] \tag{12}$$

for some functions h and g.

The example in Theorem 1 of nonTIB shape invariant demands is the special case of equations (10) (11), and (12) where $S(p) = s(p)$, $B(p, A) = b(p, A)$, $h(s, A) = -m(A)^2$, $g(x, s) = x^2$, and

$$c(s, u, A) = \left[ue^{-2m(A)s} - \frac{1}{2m(A)}\right]^{-1} + m(A). \tag{13}$$

Taken together, Theorems 1 and 2 and Corollary 1 show that although shape invariant Engel curves that do not come from TIB utility exist, the class of such models is very restrictive.

5 Convenient Testable Implications

Theorem 2 has some easily tested implications for Engel curves, described by the following corollaries. Let $w_i = \omega_i(y, A)$ denote the consumer's Engel curves for each good $i = 1, ..., n$ in some price regime.

COROLLARY 2: Let Assumption A1 hold. A necessary condition to have shape invariant Engel curves not come from TIB utility is

$$\omega_i(y, A) = h_i(A) + \lambda_i G\left[y - b(A)\right] \tag{14}$$

for some functions $h_i(A)$, $b(A)$, and $G(x)$ where G is nonlinear in x, and some constants λ_i, for goods $i = 1, ..., n$.

COROLLARY 3: Let Assumption A1 hold. Assume $\omega_i(y, A)$ is twice differentiable in y. If the consumer has shape invariant Engel curves and there exists any pair of good

i and j such that $\left[\partial\omega_i(y, A)/\partial y\right] / \left[\partial\omega_j(y, A)/\partial y\right]$ is not constant (independent of y and A), then the consumer has TIB utility.

Given shape invariance, the conditions in Corollaries 2 and 3 are sufficient to insure that utility is TIB, but they are stronger than necessary, since they do not impose the restrictions on price effects required by Theorem 2.

As noted earlier, and exemplified by equation (7), TIB utility places very few constraints on the Engel functions $G_i(x)$ in equation (1). In contrast, by Corollary 2, nonTIB shape invariance requires $G_i(x) = \lambda_i G(x)$, meaning that every good i must have the same $G_i(x)$ function up to scale. Equivalently, NonTIB shape invariance requires that for every household type A, the Engel curves for every good must be linear in the same function $G\left[y - B(A)\right]$, and it requires that all Engel curves have the same first derivatives up to scale.

Engel curves where G is linear in y are known as the Working (1943) and Leser (1963) model. Some popular demand systems have Working-Leser Engel curves, including Deaton and Muellbauer's (1980) Almost Ideal Demand System and Jorgenson, Lau, and Stoker's (1982) Indirect Translog Demand System. Such systems can satisfy the requirement for nonTIB shape invariance that $G_i(x) = \lambda_i G(x)$, but all such models must still be TIB, because NonTIB also requires that G be nonlinear.

Given semiparametric or nonparametric estimates of Engel curves $\omega_i(y, A)$, the derivative condition in Corollary 3 could be directly tested using, e.g., Haag and Hoderlein (2005) or any of the many other testing procedures cited there. One could separately nonparametrically test for shape invariance itself using Pendakur (1998). Alternatively, one could test semiparametric estimates of general shape invariant engel curves using, e.g., Blundell, Duncan, and Pendakur (1998), against semiparametric estimates of the more restrictive model of equation (14).

Extending Gorman's (1981) results on exactly aggregable demand systems, Lewbel (1991) defines the rank of any demand system (aggregable or not) as the dimension of the space spanned by its Engel curves, holding A fixed.

COROLLARY 4: Let Assumption A1 hold. All shape invariant demand systems arising from nonTIB utility functions have rank two.

Lewbel (1991) shows that the rank of a demand system equals the minimum number of functions of prices that are needed to express the cost or indirect utility function. Unlike Gorman's (1981) maximum rank three result for aggregable demands, nonaggregable demands can have any rank up to $n - 1$ where n is the number of goods. Shape invariant demand systems arising from TIB utility functions can have any rank, up to this maximum rank of $n - 1$, as can be seen by applying Lewbel (1991) to the example of equation (7) when the $\Gamma_j(x)$ functions are constructed to be linearly independent of each other. This is in sharp contrast to nonTIB shape invariant demands, which by Corollary 4 can only have rank equal to two.

The set of rank two demands is more general than the class of Engel curves given by Corollaries 2 and 3. However, most empirical studies of rank have found that consumers' demand functions have rank higher than two. See, e.g., Lewbel (1991) and Hausman, Newey, and Powell (1995). As long as rank really is greater than two, shape invariant demands must be TIB.

6 Conclusions

In empirical applications, shape invariant Engel curves, equation (1), have always been modeled as deriving from TIB utility functions, equation (4). This paper shows that exceptions exist, though they are quite restrictive. Simple tests on Engel curves can rule out these exceptions. More specific tests based on demand functions could also be constructed, essentially by semiparametrically estimating shape invariant demand functions, equation (2), and testing for the restrictions listed in Theorem 2.

In addition to convenience in model specification, these results have important implications for IB/ESE equivalence scales, which are widely used in empirical work for social welfare evaluations. TIB utility corresponds to all of the observable implications of preferences that permit the application of IB/ESE equivalence scales. If empirical demands can be shown to not satisfy the restrictions of Theorem 2, e.g., by Corollaries 2 and 4 if budget shares are shown to either be nonlinear in log income or have rank not equal to two, then shape invariance will imply TIB utility.

7 Proofs

LEMMA 1: Let Assumption A1 hold. Utility is TIB if and only if budget share demands have the form

$$w_i = \frac{\partial B(p, A)}{\partial p_i} + G_i\left[y - B(p, A), p\right] \tag{15}$$

PROOF OF LEMMA 1: If the utility function is TIB, so equation (4) holds, then equation (15) follows from equation (6) with the function G_i defined by

$$G_i\left[y - B(p, A), p\right] = -\frac{\partial V(y - B(p, A), p)/\partial p_i}{\partial V(y - B(p, A), p)/\partial y}. \tag{16}$$

For the converse, let $\widetilde{V}(p, y, A)$ be an indirect utility function associated with the demand functions in equation (15), so by Roys identity

$$-\frac{\partial \widetilde{V}(y, p, A)/\partial p_i}{\partial \widetilde{V}(y, p, A)/\partial y} = \frac{\partial B(p, A)}{\partial p_i} + G_i \left[y - B(p, A), p\right] \tag{17}$$

Now define the function V^* by $V^*(y - B(p, A), p, A) = \widetilde{V}(y, p, A)$. Strict monotonicity of $\widetilde{V}$ in y ensures existence of V^*. Rewriting equation (17) in terms of V^* gives

$$-\frac{\partial V^*(y - B(p, A), p, A)/\partial p_i}{\partial V^*(y - B(p, A), p, A)/\partial y} = G_i \left[y - B(p, A), p\right] \tag{18}$$

Define $x = y - B(p, A)$ to obtain

$$-\frac{\partial V^*(x, p, A)/\partial p_i}{\partial V^*(x, p, A)/\partial x} = G_i (x, p)$$

and define the function V by $V(x, p) = V^*(p, y, A_0)$ for any fixed reference level A_0 of A. Then

$$-\frac{\partial V(x, p)/\partial p_i}{\partial V(x, p)/\partial x} = G_i (x, p)$$

and it can then be verified that applying Roys identity to the IB indirect utility function $V(y - B(p, A), p)$ yields equation (15).

PROOF OF THEOREM 1: Applying Roys identity to the indirect utility function (8) yields

$$w_i = \frac{\partial B(p, A)}{\partial p_i} + \left[\left[y - B(p, A) - m(A)\right] 2m(A) + \left[y - B(p, A) - m(A)\right]^2\right] \frac{\partial s(p)}{\partial p_i}$$

which, after some algebraic manipulation, can be written as equation (9), which is in the form of equation (2) and hence is shape invariant. Equation (9) is also not in the form (15), so by Lemma 1 the indirect utility function (8) cannot be TIB.

PROOF OF THEOREM 2: By Shephard's lemma, $w_i = \partial C(p, u, A)/\partial p_i$, so shape invariant demands (2) have Hicksian demands of the form

$$\frac{\partial C(p, u, A)}{\partial p_i} = H_i(p, A) + G_i \left[C(p, u, A) - B(p, A), p\right] \tag{19}$$

Define $\gamma (p, u, A) = C(p, u, A) - B(p, A)$ and $h_i(p, A) = H_i(p, A) - \partial B(p, A)/\partial p_i$. Then equation (19) can be rewritten as

$$\frac{\partial \gamma (p, u, A)}{\partial p_i} = h_i(p, A) + G_i \left[\gamma (p, u, A), p\right]. \tag{20}$$

Without loss of generality, assume for every good i that $h_i(p, A_0) = 0$ for some constant reference level A_0. This is without loss of generality because if it did not hold then we could redefine $h_i(p, A)$ as $h_i(p, A) - h_i(p, A_0)$ and redefine $G_i(\gamma, p)$ as $G_i(\gamma, p) + h_i(p, A_0)$ and the demand function (20) would be unchanged. It then follows from Lemma 1 that the utility function associated with shape invariant demands (2) will not be TIB if and only if $h_i(p, A)$ is not identically zero for any function B.

Now let $x = \gamma(p, u, A)$. Taking the derivative of equation (20) with respect to p_j for any good j gives

$$\frac{\partial^2 \gamma(p, u, A)}{\partial p_i \partial p_j} = \frac{\partial h_i(p, A)}{\partial p_j} + \frac{\partial G_i\left[\gamma(p, u, A), p\right]}{\partial \gamma(p, u, A)} \frac{\partial \gamma(p, u, A)}{\partial p_j} + \frac{\partial G_i\left[\gamma(p, u, A), p\right]}{\partial p_j}$$

$$= \frac{\partial h_i(p, A)}{\partial p_j} + \frac{\partial G_i\left[\gamma(p, u, A), p\right]}{\partial \gamma(p, u, A)} \left[h_j(p, A) + G_j(\gamma(p, u, A), p)\right] + \frac{\partial G_i\left[\gamma(p, u, A), p\right]}{\partial p_j}$$

$$= \frac{\partial h_i(p, A)}{\partial p_j} + \frac{\partial G_i(x, p)}{\partial x}\left[h_j(p, A) + G_j(x, p)\right] + \frac{\partial G_i(x, p)}{\partial p_j}$$

So by Young's Theorem

$$\frac{\partial h_i(p, A)}{\partial p_j} + \frac{\partial G_i(x, p)}{\partial x}\left[h_j(p, A) + G_j(x, p)\right] + \frac{\partial G_i(x, p)}{\partial p_j} = \frac{\partial h_j(p, A)}{\partial p_i} + \frac{\partial G_j(x, p)}{\partial x}\left[h_i(p, A) + G_i(x, p)\right] + \frac{\partial Gj(x, p)}{\partial p_i}. \tag{21}$$

This is essentially Slutsky symmetry corresponding to $\gamma(p, u, A)$ instead of to the log cost function $C(p, u, A)$. Now take equation (21), and subtract from it the same expression evaluated at the reference level $A_0 \neq A$. Recalling that $h_k(p, A_0) = 0$ for $k = i, j$, this gives

$$\frac{\partial h_i(p, A)}{\partial p_j} + \frac{\partial G_i(x, p)}{\partial x} h_j(p, A) = \frac{\partial h_j(p, A)}{\partial p_i} + \frac{\partial G_j(x, p)}{\partial x} h_i(p, A). \tag{22}$$

Next take equation (22), and subtract from it the same expression evaluated at some reference level $x_0 \neq x$. This gives

$$\left[\frac{\partial G_i(x, p)}{\partial x} - \frac{\partial G_i(x_0, p)}{\partial x}\right] h_j(p, A) = \left[\frac{\partial G_j(x, p)}{\partial x} - \frac{\partial G_j(x_0, p)}{\partial x}\right] h_i(p, A). \tag{23}$$

Since x_0 is a constant, we may define $\kappa_k(p) = \partial G_k(x_0, p)/\partial x_0$ for $k = i, j$. Then equation (23) is

$$\left[\frac{\partial G_i(x, p)}{\partial x} - \kappa_i(p)\right] h_j(p, A) = \left[\frac{\partial G_j(x, p)}{\partial x} - \kappa_j(p)\right] h_i(p, A). \tag{24}$$

This equation can only hold if either $h_k(p, A) = 0$ for $k = i, j$, or

$$\frac{\partial G_k(x, p)}{\partial x} = \kappa_k(p) \quad \text{for } k = i, j \tag{25}$$

or, for some functions $g^*(x, p)$ and $h(p, A)$ that do not vary by i or j, and for some functions $\lambda_i(p)$ and $\lambda_j(p)$

$$\frac{\partial G_k(x, p)}{\partial x} - \kappa_k(p) = \lambda_k(p) g^*(x, p), \quad h_k(p, A) = \lambda_k(p) h(p, A) \quad \text{for } k = i, j \tag{26}$$

We can rule out $h_i(p, A) = 0$ for all A, since then by the argument earlier the utility function would be TIB. We can also rule out equation (25) as follows. Equation (25) requires that $G_k(x, p)$ be linear in x. Having budget shares that are linear in x requires an indirect utility functions that is ordinally equivalent to $u = e^{\theta(p,A)} \left[y - b(p, A)\right]$ for some functions θ and b (see, e.g., Muellbauer 1976 for a proof). By Roys identity the demand functions arising from this utility function have the form

$$w_i = \frac{\partial b(p, A)}{\partial p_i} + \frac{\partial \theta(p, A)}{\partial p_i} \left[y - b(p, A)\right]$$

which are shape invariant only if $\partial \theta(p, A)/\partial p_i$ does not depend on A, and in that case they are by Lemma 1 derived from TIB utility.

We are therefore left with equation (26) as the necessary conditions for shape invariant demands to not be TIB. Integrating the expression for $G_k(x, p)$ in (26) with respect to x gives, for some function $G(x, p)$ that does not vary by i or j, and for some function $\delta_k(p)$ for $k = i, j$,

$$G_k(x, p) = \lambda_k(p) G(x, p) + \kappa_k(p) x + \delta_k(p), \quad h_k(p, A) = \lambda_k(p) h(p, A) \quad \text{for } k = i, j \tag{27}$$

Subsituting equation (27) into equation (22) gives

$$\frac{\partial \left[\lambda_i(p) h(p, A)\right]}{\partial p_j} + \left(\lambda_i(p) \frac{\partial G(x, p)}{\partial x} + \kappa_i(p)\right) \left[\lambda_j(p) h(p, A)\right] = \tag{28}$$

$$\frac{\partial \left[\lambda_j(p) h(p, A)\right]}{\partial p_i} + \left(\lambda_j(p) \frac{\partial G(x, p)}{\partial x} + \kappa_j(p)\right) \left[\lambda_i(p) h(p, A)\right]$$

which simplifies to

$$\frac{\partial \lambda_i(p)}{\partial p_j} + \lambda_i(p) \left(\frac{\partial h(p, A)}{\partial p_j} \frac{1}{h(p, A)} - \kappa_j(p)\right) = \frac{\partial \lambda_j(p)}{\partial p_i} + \lambda_j(p) \left(\frac{\partial h(p, A)}{\partial p_i} \frac{1}{h(p, A)} - \kappa_i(p)\right). \tag{29}$$

Difference this with respect to A, using $h(p, A_0) = 0$ to obtain

$$\lambda_i(p)\left(\frac{\partial h(p, A)}{\partial p_j}\frac{1}{h(p, A)}\right) = \lambda_j(p)\left(\frac{\partial h(p, A)}{\partial p_i}\frac{1}{h(p, A)}\right). \tag{30}$$

Subtracting (30) from (29) gives

$$\frac{\partial \lambda_i(p)}{\partial p_j} - \lambda_i(p)\kappa_j(p) = \frac{\partial \lambda_j(p)}{\partial p_i} - \lambda_j(p)\kappa_i(p) \tag{31}$$

Now observe that for any function $\sigma(p)$ and any nonzero function $\tau(p)$ we may define $\overline{g}(x, p) = \left[G(x, p) - \sigma(p)\right]/\tau(p)$ and by equation (27) we obtain, for $k = i, j$,

$$G_k(x, p) = \lambda_k(p)\left[\tau(p)\overline{g}(x, p) + \sigma(p)\right] + \kappa_k(p)x + \delta_k(p). \tag{32}$$

Let $\widetilde{g}(u, p) = \overline{g}\left[\gamma(p, u, A_0), p\right]$. Evaluate equation (20) at $A = A_0$, substituting in equation)(32 to get

$$\frac{\partial \gamma(p, u, A_0)}{\partial p_k} = \lambda_k(p)\left[\tau(p)\widetilde{g}(u, p) + \sigma(p)\right] + \kappa_k(p)\gamma(p, u, A_0) + \delta_k(p). \tag{33}$$

Take the derivative of this expression with respect to u, and divide the result by $\partial\gamma(p, u, A_0)/\partial u$ to get

$$\frac{\partial \ln\left[\partial\gamma(p, u, A_0)/\partial u\right]}{\partial p_k} = \lambda_k(p)\tau(p)\varsigma(p, u) + \kappa_k(p). \tag{34}$$

where

$$\varsigma(p, u) = \frac{\partial\widetilde{g}(u, p)/\partial u}{\partial\gamma(p, u, A_0)/\partial u} \tag{35}$$

noting that $\partial\gamma(p, u, A_0)/\partial u > 0$ because $\partial\gamma(p, u, A_0)/\partial u = \partial C(p, u, A_0)/\partial u$ for cost function C. By equation (34),

$$\frac{\partial\left[\ln\left[\partial\gamma(p, u_1, A_0)/\partial u\right] - \ln\left[\partial\gamma(p, u_0, A_0)/\partial u\right]\right]}{\partial p_k} = \lambda_k(p)\tau(p)\left[\varsigma(p, u_1) - \varsigma(p, u_0)\right]. \tag{36}$$

for any utility levels u_1 and u_0. Now $\varsigma(p, u_1) - \varsigma(p, u_0)$ must be nonzero for some u_1 and u_0, since otherwise for some function $\widetilde{r}(p)$

$$\frac{\partial\widetilde{g}(u, p)}{\partial u} = \widetilde{r}(p)\frac{\gamma(p, u, A_0)}{\partial u}$$

which requires for some function $\widetilde{s}(p)$ that $\widetilde{g}(u, p) = \widetilde{s}(p) + \widetilde{r}(p)\gamma(p, u, A_0)$, so $\overline{g}(x, p) = \widetilde{s}(p) + \widetilde{r}(p)x$, which we can rule out because it implies G_k is linear in x and so comes from TIB utility.

Equation (36) holds for any choice of functions $\tau(p) \neq 0$, so let

$$\tau(p) = \left[\varsigma(p, u_1) - \varsigma(p, u_0)\right]^{-1}$$

Also, define the function $S(p)$ by

$$S(p) = \ln\left[\partial\gamma(p, u_1, A_0)/\partial u\right] - \ln\left[\partial\gamma(p, u_0, A_0)/\partial u\right]$$

With these definitions, equation (36) becomes

$$\frac{\partial S(p)}{\partial p_k} = \lambda_k(p) \tag{37}$$

for $k = i, j$. Substituting equation (37) into equation (31) gives

$$\frac{\partial^2 S(p)}{\partial p_i \partial p_j} - \frac{\partial S(p)}{\partial p_i}\kappa_j(p) = \frac{\partial^2 S(p)}{\partial p_j \partial p_i} - \frac{\partial S(p)}{\partial p_i}\kappa_i(p)$$

which requires that, for some function $t(p)$,

$$\kappa_k(p) = \frac{\partial S(p)}{\partial p_k} t(p) \tag{38}$$

Substituting equations (37) and (38) into equation (33) gives

$$\frac{\partial\gamma(p, u, A_0)}{\partial p_k} = \frac{\partial S(p)}{\partial p_k}\left[\tau(p)\widetilde{g}(u, p) + \gamma(p, u, A_0) + \sigma(p)\right] + \delta_k(p). \tag{39}$$

Equation (39) holds for any choice of function $\sigma(p)$, so define

$$\sigma(p) = -\left[\tau(p)\widetilde{g}(u_0, p) + \gamma(p, u_0, A_0)\right]$$

and define the function $R(p)$ by $R(p) = \gamma(p, u_0, A_0)$. It then follows from equation (39) that

$$\frac{\partial R(p)}{\partial p_k} = \delta_k(p). \tag{40}$$

Substituting (37), (38) and (40) into (32) gives

$$G_k(x, p) = \frac{\partial S(p)}{\partial p_k}\left[\tau(p)\left[\overline{g}(x, p) + x\right] + \sigma(p)\right] + \frac{\partial R(p)}{\partial p_k}. \tag{41}$$

Now define $g(x, p) = \tau(p)\left[\overline{g}(x, p) + x\right] + \sigma(p)$, and by equations (37) and (41), equation (27) becomes

$$G_k(x, p) = \frac{\partial S(p)}{\partial p_k} g(x, p) + \frac{\partial R(p)}{\partial p_k}, \quad h_k(p, A) = \frac{\partial S(p)}{\partial p_k} h(p, A) \quad \text{for } k = i, j. \tag{42}$$

Now substitute equation (42) into equation (20) to obtain

$$\frac{\partial\left[\gamma(p,u,A)-R(p)\right]}{\partial p_i}=\frac{\partial S(p)}{\partial p_i}\left[h(p,A)+g(\gamma(p,u,A),p)\right]. \tag{43}$$

Recall that $\gamma(p,u,A)$ was defined by $\gamma(p,u,A)=C(p,u,A)-B(p,A)$. Without loss of generality we may therefore assume $R(p)=0$, since if $R(p)\neq 0$ we could have started out by redefining $B(p,A)$ as $B(p,A)+R(p)$. Equation (43) then reduces to

$$\frac{\partial\gamma(p,u,A)}{\partial p_i}=\frac{\partial S(p)}{\partial p_i}\left[h(p,A)+g(\gamma(p,u,A),p)\right]. \tag{44}$$

Consider taking the total derivative of $\gamma(p,u,A)$ with respect to p, holding u and A fixed. Using equation (44),

$$\begin{aligned}\frac{d\gamma(p,u,A)}{dS(p)} &= \sum_i \frac{\partial\gamma(p,u,A)}{\partial p_i}\frac{dp_i}{dS(p)}\\ &= \sum_i \frac{\partial S(p)}{\partial p_i}\frac{dp_i}{dS(p)}\left[h(p,A)+g(\gamma(p,u,A),p)\right]\\ &= \frac{dS(p)}{dS(p)}\left[h(p,A)+g(\gamma(p,u,A),p)\right]\end{aligned}$$

so

$$\frac{d\gamma(p,u,A)}{dS(p)}=h(p,A)+g(\gamma(p,u,A),p) \tag{45}$$

which, substituted into equation (44) gives

$$\frac{\partial\gamma(p,u,A)}{\partial p_i}=\frac{d\gamma(p,u,A)}{dS(p)}\frac{\partial S(p)}{\partial p_i}$$

and therefore

$$\gamma(p,u,A)=c\left[S(p),u,A\right]$$

for some function c. With equation (45) this implies

$$\frac{dc\left[S(p),u,A\right]}{dS(p)}=h(p,A)+g\left[c\left(S(p),u,A\right),p\right].$$

This shows that the functions h and g can depend on p only through $S(p)$, so without loss of generality we may rewrite these functions as $h\left[S(p),A\right]$ and $g\left[x,S(p)\right]$, so

$$\frac{dc\left[S(p),u,A\right]}{dS(p)}=h\left[S(p),A\right]+g\left[c\left(S(p),u,A\right),S(p)\right]. \tag{46}$$

Equation (10) then follows from the definitions of c, γ , B and ω_i.

PROOF OF COROLLARY 1: Equation (46) shows that the function c must be a solution to the differential equation (12), and the cost function follows from the definitions of c, γ , B.

PROOF OF COROLLARY 2: Let $p = p_0$ denote the price regime. Then equation (14) follows from equation (10) with $b(A) = B(p_0, A)$, $Gy - B(A) = gy - B(A, p_0)$, $\lambda_i = \partial S(p_0)/\partial p$, and

$$h_i(A) = \frac{\partial B(p, A)}{\partial p_i} + \frac{\partial S(p)}{\partial p_i} h(p, A) + \delta_i(p).$$

PROOF OF COROLLARY 3: If the consumer has shape invariance but does not have TIB utility, then by equation (14), $\left[\partial\omega_i(y, A)/\partial y\right] / \left[\partial\omega_j(y, A)/\partial y\right] = \lambda_i/\lambda_j$, which is independent of y and A.

PROOF OF COROLLARY 4: By Corollary 1, the cost functions of all nonTIB shape invariant demand systems depends on p only through two functions, S and B. Theorem 1 of Lewbel (1991) shows that the rank of a demand system equals the minimum number of functions of prices that can be used to write the cost function, so the rank of all nonTIB shape invariant demand systems must be less than or equal to two. They cannot be rank one, because Lewbel (1991) shows that all rank one demand systems are homothetic. All homothetic demand systems are special cases of demands that are linear in x, and by Theorem 1, nonTIB shape invariant demands cannot be linear in x.

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